Journal of Statistical Physics, Vol. 99, Nos. 5/6, 2000

On Condensations in the Bogoliubov Weakly Imperfect Bose Gas

J.-B. Bru[1,2,4] **and V. A. Zagrebnov**[1,3,4]

Received June 9, 1999; final November 10, 1999

We show that condensation in the Bogoliubov weakly imperfect Bose gas (WIBG) may appear in two stages. If interaction is such that the pressure of the WIBG does not coincide with the pressure of the perfect Bose gas (PBG), then the WIBG may manifest two kinds of condensations: *nonconventional* Bose condensation in zero mode, due to the interaction (the first stage), and *conventional* (generalized) Bose–Einstein condensation in modes next to the zero mode due to the particle density saturation (the second stage). Otherwise the WIBG manifests only the latter kind of condensation.

KEY WORDS: Bogoliubov weakly imperfect gas; Bose condenstation; generalized condensation; conventional; nonconventional.

1. INTRODUCTION AND SETUP OF THE PROBLEM

To fix the notation we recall some facts about the Bogoliubov Weakly Imperfect Bose Gas. Thus we consider a system of bosons of mass m enclosed in a cubic box $\Lambda = L \times L \times L \subset \mathbb{R}^{d=3}$ of volume $V \equiv |\Lambda| = L^3$. If the particle interaction is defined by a translation-invariant absolutely integrable two-body potential $\varphi(x)$ and

$$v(q) = \int_{\mathbb{R}^3} d^3x\, \varphi(x)\, e^{-iqx}, \qquad q \in \mathbb{R}^3 \tag{1.1}$$

[1] Instituut voor Theoretische Fysica, K. U. Leuven, Belgium.
[2] Allocataire de Recherche MRT; e-mail: bru@cpt.univ-mrs.fr.
[3] Université de la Méditerranée (Aix-Marseille II); e-mail: zagrebnov@cpt.univ-mrs.fr.
[4] On leave of absence from Centre de Physique Théorique (Unité Propre de Recherche 7061), CNRS-Luminy-Case 907; 13288 Marseille, Cedex 09, France.

then assuming periodic boundary conditions on $\partial\Lambda$, the Hamiltonian of the system acting on the boson Fock space $\mathscr{F}_\Lambda$ can be written in the second quantized form as

$$H_\Lambda = \sum_{k \in \Lambda^*} \varepsilon_k a_k^* a_k + \frac{1}{2V} \sum_{k_1, k_2, q \in \Lambda^*} v(q)\, a_{k_1+q}^* a_{k_2-q}^* a_{k_1} a_{k_2} \tag{1.2}$$

where the sums run over the set

$$\Lambda^* = \left\{ k \in \mathbb{R}^3 : \alpha = 1, 2, 3,\, k_\alpha = \frac{2\pi n_\alpha}{L} \text{ and } n_\alpha = 0, \pm 1, \pm 2, \ldots \right\} \tag{1.3}$$

Here $\varepsilon_k = \hbar^2 k^2/2m$ is free-particle spectrum, and $a_k^\# = \{a_k^*, a_k\}$ are usual boson creation and annihilation operators in the one-particle state $\psi_k(x) = V^{-1/2} e^{ikx}$, $k \in \Lambda^*$, $x \in \Lambda$. For example, $a_k^* \equiv a^*(\psi_k) = \int_\Lambda dx\, \psi_k(x)\, a^*(x)$ where $a^\#(x)$ are basic boson operators on the boson Fock space $\mathscr{F}_\Lambda$ over $L^2(\Lambda)$.

Throughout this paper we suppose that:

(A) $\varphi(x) = \varphi(\|x\|)$ and $\varphi \in L^1(\mathbb{R}^3)$;

(B) $v(k)$ is a real continuous function, satisfying $v(0) > 0$ and $0 \leqslant v(k) \leqslant v(0)$ for $k \in \mathbb{R}^3$.

If one supposes that the Bose–Einstein condensation, which occurs for $\varphi(x) \equiv 0$ (Perfect Bose Gas (PBG)) in the mode $k = 0$, persists for a weak interaction $\varphi(x)$, then according to Bogoliubov[1, 2] the most important interaction terms in (1.2) should be those which contain at least two zero-mode operators $a_0^\#$. Thus we come to the following truncated Hamiltonian (Bogoliubov Hamiltonian for Weakly Imperfect Bose Gas (WIBG), see refs. 1, 2):

$$H_\Lambda^B = T_\Lambda + U_\Lambda^D + U_\Lambda \tag{1.4}$$

where

$$T_\Lambda = \sum_{k \in \Lambda^*} \varepsilon_k a_k^* a_k \tag{1.5}$$

$$U_\Lambda^D = \frac{v(0)}{V} a_0^* a_0 \sum_{k \in \Lambda^*, k \neq 0} a_k^* a_k + \frac{1}{2V} \sum_{k \in \Lambda^*, k \neq 0} v(k)\, a_0^* a_0 (a_k^* a_k + a_{-k}^* a_{-k}) + \frac{v(0)}{2V} a_0^{*2} a_0^2 \tag{1.6}$$

$$U_\Lambda = \frac{1}{2V} \sum_{k \in \Lambda^*, k \neq 0} v(k) (a_k^* a_{-k}^* a_0^2 + a_0^{*2} a_k a_{-k}) \tag{1.7}$$

The Hamiltonian was aimed to extract the Landau gapless spectrum of excitations which implies the well-known microscopic theory of superfluidity.[1,2] To realize this program Bogoliubov proposed the approximation: $a_0^{\#}/\sqrt{V} \to c^{\#}$, reducing the Hamiltonian (1.4) to a bilinear form, which can be diagonalized to give this spectrum explicitly, see Appendix A.

Therefore, to verify this concept one has to study the WIBG refraining from the Bogoliubov approximation. The first rigorous result in this direction[3] shows that thermodynamic properties of the WIBG are different from predictions of the Bogoliubov theory. Exact solution[4,5] of the model (1.4) proves that the Bogoliubov approximation $a_0^{\#}/\sqrt{V} \to c^{\#}$ in (1.4) eliminates quantum fluctuations of operators $a_0^{\#}/\sqrt{V}$. In fact it is important to retain these fluctuations since they are a cause of effective *attraction* between bosons in the mode $k=0$, see ref. 6. In contrast to standard (we call it *conventional*) Bose–Einstein condensation in the PBG (which is due to saturation of the grand-canonical particle density when the chemical potential $\mu \nearrow 0$ for $d \geqslant 3$), it is this effective attraction which is now responsible for accumulation of a macroscopic number of bosons in the zero-mode. We call it *nonconventional* (Bose) condensation, see Appendix B for classification of different Bose condensations. In refs. 4, 5 we pointed out sufficient and necessary conditions on the potential (1.1) which imply the *nonconventional* condensation in the WIBG. For reader's convenience, we resume below the thermodynamic properties of the WIBG.

Papers[4,5] show that the pressure of the Bogoliubov WIBG can be calculated exactly in the thermodynamic limit:

Proposition 1.1. The pressure $p_\Lambda^B(\beta,\mu)$ associated with the Bogoliubov Hamiltonian H_Λ^B, i.e.,

$$p_\Lambda^B(\beta,\mu) \equiv p_\Lambda[H_\Lambda^B] \equiv \frac{1}{\beta V} \ln Tr_{\mathscr{F}_\Lambda} e^{-\beta(H_\Lambda^B - \mu N_\Lambda)} \tag{1.8}$$

is defined only in domain $Q=\{\mu \leqslant 0\} \times \{\theta \geqslant 0\}$ and it is equal (in the thermodynamic limit) to

$$p^B(\beta,\mu) = \sup_{c \in \mathbb{C}} \tilde{p}^B(\beta,\mu;c^{\#}) \equiv \lim_{\Lambda} \{\sup_{c \in \mathbb{C}} \tilde{p}_\Lambda^B(\beta,\mu;c^{\#})\} \tag{1.9}$$

with $\tilde{p}_\Lambda^B(\beta,\mu;c^{\#})$ defined by (A.4) in Appendix A, $c^{\#}=(c \text{ or } \bar{c})$. Here

$$N_\Lambda = \sum_{k \in \Lambda^*} N_k \equiv \sum_{k \in \Lambda^*} a_k^* a_k$$

is the particle-number operator, μ and $\theta = \beta^{-1}$ are respectively chemical potential and temperature of the system in the grand-canonical ensemble.

Corollary 1.2. Let $v(k)$ satisfy (A), (B) and

$$v(0) \geqslant \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} d^3k \, \frac{v(k)^2}{\varepsilon_k} \tag{1.10}$$

Then

$$p^B(\beta, \mu) = \sup_{c \in \mathbb{C}} \tilde{p}^B(\beta, \mu; c^{\#}) = \tilde{p}^B(\beta, \mu; 0) = p^P(\beta, \mu) \tag{1.11}$$

where

$$p^P(\beta, \mu) \equiv \lim_{\Lambda} p_\Lambda[T_\Lambda]$$

is the grand-canonical pressure of the PBG.

Corollary 1.3. Let $v(k)$ satisfy (A), (B) and (C):

$$v(0) < \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} d^3k \, \frac{v(k)^2}{\varepsilon_k} \tag{1.12}$$

Then there are $\mu_0 < 0$ and $\theta_0(\mu) > 0$ such that one has

$$p^B(\beta, \mu) = \sup_{c \in \mathbb{C}} \tilde{p}^B(\beta, \mu; c^{\#}) = \tilde{p}^B(\beta, \mu; \hat{c}^{\#}(\theta, \mu) \neq 0) > p^P(\beta, \mu) \tag{1.13}$$

for $(\theta, \mu) \in D$ defined by

$$D = \{(\theta, \mu) : \mu_0 < \mu \leqslant 0, 0 \leqslant \theta < \theta_0(\mu)\} \tag{1.14}$$

and

$$p^B(\beta, \mu) = \sup_{c \in \mathbb{C}} \tilde{p}^B(\beta, \mu; c^{\#}) = p^P(\beta, \mu) \tag{1.15}$$

for $(\theta, \mu) \notin \bar{D}$.

Remark 1.4. Notice that the condition (C) is trivially implemented for low dimensions $d = 1, 2$.

The inequality (1.13) gives access to analysis of the macroscopic occupation of the zero-mode which is exclusively due to the WIBG interaction.

Proposition 1.5. D is a domain which corresponds to nonconventional condensation in the mode $k=0$:

$$\rho_0^B(\theta,\mu) \equiv \lim_{\Lambda} \left\langle \frac{a_0^* a_0}{V} \right\rangle_{H_\Lambda^B} (\beta,\mu)$$
$$= \begin{Bmatrix} |\hat{c}(\theta,\mu)|^2 > 0, \ (\theta,\mu) \in D \\ 0, \ (\theta,\mu) \in Q \backslash \bar{D} \end{Bmatrix} \tag{1.16}$$

Here $\hat{c}(\theta,\mu)$ is defined by (1.13) and

$$\omega_\Lambda^B(-) \equiv \langle - \rangle_{H_\Lambda^B} (\beta,\mu) \tag{1.17}$$

represents the grand-canonical Gibbs state for the Hamiltonian H_Λ^B. We showed that the *nonconventional* Bose-condensate (1.16) undergoes a jump on the boundary ∂D, see ref. 4, 5 and Fig. 1.

However, we have to admit that in refs. 4, 5 we studied the WIBG only in the grand-canonical ensemble, i.e., by fixing the chemical potential μ and the temperature $\theta = \beta^{-1}$. On the other hand, it is well-known that the *conventional* Bose–Einstein condensation appears only due to the Bose-statistics and saturation of the particle density when $\mu \nearrow 0$ (Appendix B). For example, the PBG condensate density $\rho_0^P(\theta)$ is parameterized by total particle density ρ which should be higher than the critical particle density $\rho_c^P(\theta) \equiv \rho^P\ (\theta, \mu = 0)$, corresponding to the saturation at $\mu = 0$:

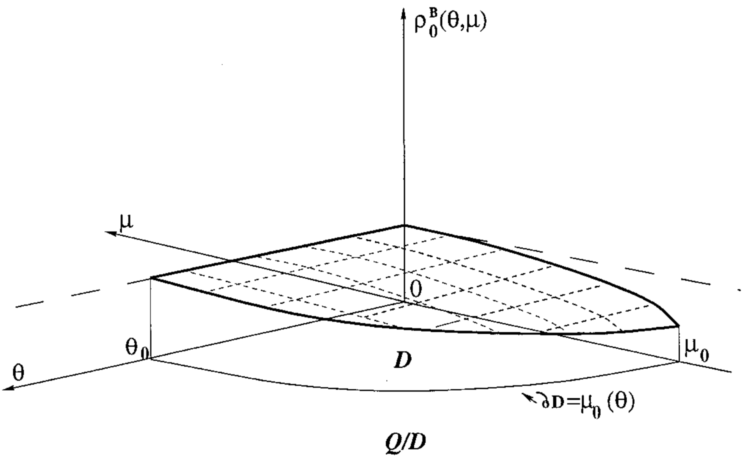

Fig. 1. Illustration of the nonconventional condensate density $\rho_0^B(\theta,\mu)$ as a function of the chemical potential μ and the temperature θ for the model H_Λ^B.

$\rho_0^P(\theta) \equiv \rho - \rho_c^P(\theta)$. Here $\rho^P(\theta, \mu)$ is the total particle density of the PBG in the grand-canonical ensemble.

Thus the aim of the present paper is to study thermodynamic properties of the WIBG and *conventional* Bose–Einstein condensation as a function of the total particle density ρ. In particular, we show that similar to the PBG there is a critical particle density $\rho_c^B(\theta)$ such that for densities $\rho > \rho_c^B(\theta)$ there exists a *conventional* Bose condensation $\tilde{\rho}_0^B(\theta) = \rho - \rho_c^B(\theta)$. It has nothing to do with the *nonconventional* Bose condensation: if $\theta \leqslant \theta_0(0)$ this *conventional* condensation appears *after* the *nonconventional* one (1.16), see Fig. 1. This main statements is formulated in the next Section 2. Since the *conventional* Bose–Einstein condensation could be of different types (see Appendix B), we make there also its classification. In Section 3 we study the Bogoliubov approximation for the state $\hat{\omega}_\Lambda^B$ à la Ginibre.[7] This allows to compare it (in the infinite volume limit) with the corresponding limit of the state ω_Λ^B (1.17). Section 4 summarizes the results and contains additional remarks. Some definitions and technical statements are formulated in Appendices A–E.

2. CONVENTIONAL BOSE–EINSTEIN CONDENSATION IN THE WIBG

First we establish that (similar to the PBG) the particle density $\rho^B(\theta, \mu)$ of the WIBG is saturated when $\mu \nearrow 0$, i.e., there exists a critical particle density $\rho_c^B(\theta) = \lim_{\mu \to 0^-} \rho^B(\theta, \mu)$. Indeed, using the Griffiths Lemma (see refs. 8, 9 or Appendix C) and the Propositions 1.1 and 1.5, one finds for the grand-canonical total particle density in the WIBG:

$$\begin{aligned}\rho^B(\theta, \mu) &\equiv \lim_\Lambda \omega_\Lambda^B\left(\frac{N_\Lambda}{V}\right) = \lim_\Lambda \frac{1}{V} \sum_{k \in \Lambda^*} \omega_\Lambda^B(N_k) \\ &= \lim_\Lambda \partial_\mu p_\Lambda^B(\beta, \mu) = \partial_\mu \tilde{p}^B(\beta, \mu; 0) \\ &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} (e^{\beta(\varepsilon_k - \mu)} - 1)^{-1}\, d^3k \end{aligned} \tag{2.1}$$

for $(\theta, \mu < 0) \in Q \backslash \bar{D}$, whereas for $(\theta, \mu < 0) \in D$ one has:

$$\begin{aligned}\rho^B(\theta, \mu) &= \partial_\mu \tilde{p}^B(\beta, \mu; \hat{c}^{\#}(\theta, \mu) \neq 0) \\ &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \left[\frac{f_k}{E_k} (e^{\beta E_k} - 1)^{-1} + \frac{h_k^2}{2E_k(f_k + E_k)}\right]_{c = \hat{c}(\theta, \mu)} d^3k \\ &\quad + |\hat{c}(\theta, \mu)|^2 \end{aligned} \tag{2.2}$$

Then, from (2.1) and (2.2), we see that the total particle density $\rho^B(\theta, \mu)$ reaches its maximal (critical) value $\rho_c^B(\theta) \equiv \rho^B(\theta, \mu = 0)$ at $\mu = 0$:

(i) for $\theta > \theta_0$ $(\mu = 0)$ one gets

$$\rho_c^B(\theta) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} (e^{\beta \varepsilon_k} - 1)^{-1} \, d^3k = \rho_c^P(\theta) < +\infty \tag{2.3}$$

(ii) for $\theta < \theta_0$ $(\mu = 0)$ one has

$$\rho_c^B(\theta) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \left[\frac{f_k}{E_k} (e^{\beta E_k} - 1)^{-1} + \frac{h_k^2}{2E_k(f_k + E_k)} \right]_{c = \hat{c}(\theta, 0),\, \mu = 0} d^3k + |\hat{c}(\theta, \mu = 0)|^2 < +\infty \tag{2.4}$$

by virtue of the estimate $|\hat{c}(\theta, \mu = 0)|^2 \leqslant const$, see ref. 5. By convexity of $p^B(\beta, \mu)$ with respect to μ, one gets that the function $\rho^B(\theta, \mu)$ is monotonous and

$$\lim_{\mu \to \mu_0(\theta)^-} \rho^B(\theta, \mu) \equiv \rho_{\inf}^B(\theta) < \lim_{\mu \to \mu_0(\theta)^+} \rho^B(\theta, \mu) \equiv \rho_{\sup}^B(\theta) \tag{2.5}$$

where $\mu_0(\theta)$ is the inverse function of $\theta_0(\mu)$, see (1.14), and

$$\lim_{\theta \to \theta_0(0)^+} \rho_c^B(\theta) < \lim_{\theta \to \theta_0(0)^-} \rho_c^B(\theta) \tag{2.6}$$

The total density $\rho^B(\theta, \mu)$ is illustrated by Fig. 2.

Now we study the WIBG for temperatures and total particle densities as given parameters. By Lemma D.1 of Appendix D, there exists $\varepsilon_{\Lambda,1} > 0$ such that for $\mu < \varepsilon_{\Lambda,1} < \varepsilon_{\|k\| = 2\pi/L}$,

$$\omega_\Lambda^B \left(\frac{N_\Lambda}{V} \right) < +\infty$$

although

$$\lim_{\mu \to \varepsilon_{\Lambda,1}} \omega_\Lambda^B \left(\frac{N_\Lambda}{V} \right) = +\infty \tag{2.7}$$

Therefore, for any $\rho > 0$, there is a unique value of the chemical potential $\mu_\Lambda^B(\theta, \rho) < \varepsilon_{\Lambda,1}$ such that

$$\left\langle \frac{N_\Lambda}{V} \right\rangle_{H_\Lambda^B} (\beta, \mu_\Lambda^B(\theta, \rho)) = \omega_\Lambda^B \left(\frac{N_\Lambda}{V} \right) = \rho \tag{2.8}$$

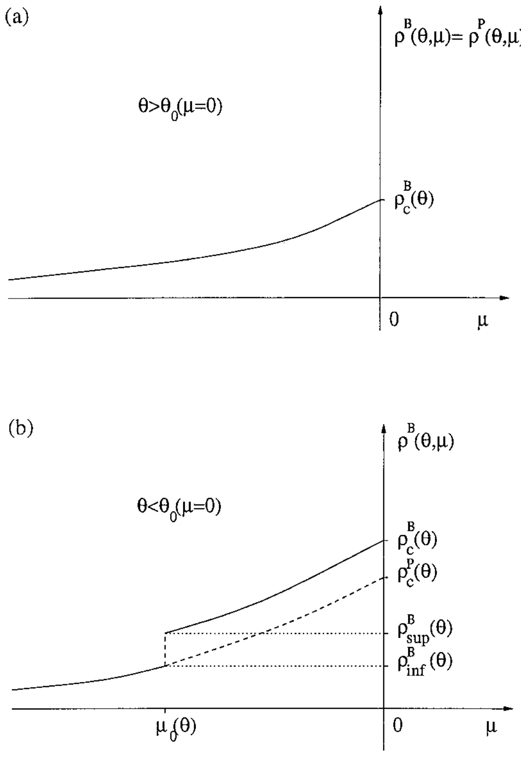

Fig. 2. Illustration of the total particle density as a function of the chemical potential μ for the model H_Λ^B at fixed temperature $\theta = \beta^{-1}$: (a) if $\theta > \theta_0(0)$: the graph of $\rho^B(\theta, \mu) = \rho^P(\theta, \mu)$, where $\rho_c^P(\theta)$ is the total density for the PBG (notice that $\rho_c^B(\theta) = \rho_c^P(\theta) \equiv \rho^P(\theta, 0)$); (b) if $\theta < \theta_0(0)$: the graph of $\rho^B(\theta, \mu) \geqslant \rho^P(\theta, \mu)$ (notice that in this case $\rho_c^B(\theta) > \rho_c^P(\theta)$).

Notice that for $\rho < \rho_c^B(\theta)$ the monotonicity of $\rho^B(\theta, \mu)$ for $\mu \leqslant 0$ implies that (2.8) has a unique solution

$$\mu^B(\theta, \rho) = \lim_{\Lambda} \mu_\Lambda^B(\theta, \rho) < 0$$

independent of the presence of the *nonconventional* condensation, see (2.1) and (2.2). Therefore, below the saturation limit $\rho_c^B(\theta)$ one has at most the *nonconventional* condensation (1.16) known from refs. 4, 5. In the rest of this section we consider the case $\rho \geqslant \rho_c^B(\theta)$. In general, for any $\rho \geqslant \rho_c^B(\theta)$, one gets by (2.7) and (2.8) that $\mu_\Lambda^B(\theta, \rho) \gtrless 0$ and

$$\lim_{\Lambda} \mu_\Lambda^B(\theta, \rho \geqslant \rho_c^B(\theta)) = 0 \tag{2.9}$$

From now on we set

$$\omega^B_{\Lambda,\rho}(-) \equiv \omega^B_\Lambda(-)\,|_{\mu=\mu^B_\Lambda(\theta,\rho)} \tag{2.10}$$

According to refs. 4, 5 the WIBG nonconventional condensation in the mode $k=0$ is saturated for $\mu \to 0^-$ either by $|\hat{c}(\theta,0)|^2 > 0$ (for $\theta < \theta_0(0)$), or by $|\hat{c}(\theta,0)|^2 = 0$ (for $\theta > \theta_0(0)$), see (1.16) and Fig. 1. Therefore, by (2.1)–(2.4) and Lemma D.1 the saturation of the total particle density $\rho^B(0,\mu)$ for $\mu \to 0^-$ should imply a *conventional* Bose condensation in modes next to $k=0$ (for discussion of coexistence of these *two kinds* of condensations in the framework of simplified models see e.g., recent papers[10, 11]).

To control the condensation for $k \neq 0$ we introduce an auxiliary Hamiltonian

$$H^B_{\Lambda,\alpha} = H^B_\Lambda - \alpha \sum_{\{k\in\Lambda^*,\, a<\|k\|<b\}} a^*_k a_k$$

for $0<a<b$. We set

$$p^B_\Lambda(\beta,\mu,\alpha) \equiv \frac{1}{\beta V} \ln Tr_{\mathscr{F}_\Lambda} e^{-\beta H^B_{\Lambda,\alpha}(\mu)} \tag{2.11}$$

and

$$\omega^{B,\alpha}_\Lambda(-) \equiv \langle - \rangle_{H^B_{\Lambda,\alpha}}(\beta,\mu)$$

for the grand-canonical Gibbs state corresponding to $H^B_{\Lambda,\alpha}(\mu)$.

Recall that by (1.14) $\mu_0(\theta)$ is the function (inverse to $\theta_0(\mu)$) which defines a borderline of domain D, see Fig. 1.

Proposition 2.1. Let $\alpha \in [-\delta,\delta]$ where $0 \leqslant \delta \leqslant \varepsilon_a/2$ and $\varepsilon_a = \inf_{\|k\|\geqslant a} \varepsilon_k$. Then there exists a domain $D_\delta \subset D$:

$$D_\delta \equiv \{(\theta,\mu) : \mu_0 < \mu_0(\delta) \leqslant \mu \leqslant 0,\, 0 \leqslant \theta \leqslant \theta_0(\mu,\delta) < \theta_0(\mu)\} \tag{2.12}$$

(see Fig. 3) such that

$$|p^B_\Lambda(\beta,\mu,\alpha) - \sup_{c\in\mathbb{C}} \tilde{p}^B_\Lambda(\beta,\mu,\alpha;c^{\#})| \leqslant \frac{K(\delta)}{\sqrt{V}} \tag{2.13}$$

for V sufficiently large, uniformly in $\alpha \in [-\delta,\delta]$ and for:

$$\text{(i)} \quad (\theta,\mu) \in D_\delta, \qquad \text{if} \quad \mu^B_\Lambda(\theta, \rho \geqslant \rho^B_c(\theta)) \leqslant 0$$

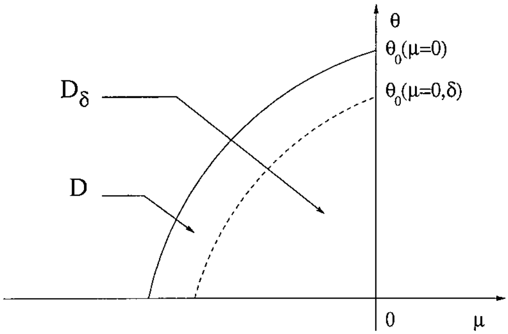

Fig. 3. Illustration of the domain $D_\delta \subset D$.

or

$$\text{(ii)} \quad (\theta, \mu) \in D_\delta \cup \{(\theta, \mu) : 0 \leqslant \mu \leqslant \mu_\Lambda^B(\theta, \rho \geqslant \rho_c^B(\theta)), 0 \leqslant \theta \leqslant \theta_0(\mu = 0, \delta)\} \tag{2.14}$$

if $\mu_\Lambda^B(\theta, \rho \geqslant \rho_c^B(\theta)) \geqslant 0$.

Proof. The existence of the domain D_δ follows from the proof of Theorem 3.14.[5] This means that the estimate (2.13) is stable with respect to local perturbations of the free-particle spectrum: $\varepsilon_k \to \varepsilon_k - \alpha\chi_{(a,b)}(\|k\|)$, for $|\alpha| \leqslant \delta \leqslant \varepsilon_a/2$ in a reduced domain $D_\delta \subset D$. Here $\chi_{(a,b)}(\|k\|)$ is the characteristic function of interval $(a, b) \subset \mathbb{R}$. Extension in (2.14) is due to continuity of the pressure $p_\Lambda^B(\beta, \mu, \alpha)$ and the trial pressure $\tilde{p}_\Lambda^B(\beta, \mu, \alpha; c^\#)$ in parameters $\alpha \in [-\delta, \delta]$ and $\mu \leqslant \mu_\Lambda^B(\theta, \rho \geqslant \rho_c^B(\theta))$, see (2.8), (2.9). ∎

Corollary 2.2. Let $\rho \geqslant \rho_c^B(\theta)$, see (2.3), (2.4). Then for $\theta < \theta_0(0)$ one has

$$\lim_\Lambda \frac{1}{V} \sum_{\{k \in \Lambda^*, a < \|k\| < b\}} \omega_{\Lambda,\rho}^B(N_k) = \frac{1}{(2\pi)^3} \int_{a < \|k\| < b} \left[\frac{f_k}{E_k} (e^{\beta E_k} - 1)^{-1} + \frac{h_k^2}{2E_k(f_k + E_k)} \right]_{c = \hat{c}(\theta, 0), \mu = 0} d^3k \tag{2.15}$$

whereas for $\theta > \theta_0(0)$ one gets

$$\lim_\Lambda \frac{1}{V} \sum_{\{k \in \Lambda^*, a < \|k\| < b\}} \omega_{\Lambda,\rho}^B(N_k) = \frac{1}{(2\pi)^3} \int_{a < \|k\| < b} (e^{\beta \varepsilon_k} - 1)^{-1} d^3k \tag{2.16}$$

Proof. Consider the sequence of functions $\{p_\Lambda^B(\beta, \mu_\Lambda^B(\theta, \rho), \alpha)\}_\Lambda$ defined by (2.11), where chemical potential is a solution of (2.8), for the corresponding Hamiltonian and $\alpha \in [-\delta, \delta]$. Since by (2.11)

$$\partial_\alpha p_\Lambda^B(\beta, \mu_\Lambda^B(\theta, \rho), \alpha) = \frac{1}{V} \sum_{\{k \in \Lambda^*,\, a < \|k\| < b\}} \omega_{\Lambda, \rho}^{B, \alpha}(N_k) \tag{2.17}$$

and $\{p_\Lambda^B(\beta, \mu_\Lambda^B(\theta, \rho), \alpha)\}_\Lambda$ are convex functions of $\alpha \in [-\delta, \delta]$, Proposition 2.1 and the Griffiths Lemma (refs. 8, 9 or Appendix C) imply

$$\begin{aligned}\lim_\Lambda \partial_\alpha p_\Lambda^B(\beta, \mu_\Lambda^B(\theta, \rho), \alpha) &= \lim_\Lambda \frac{1}{V} \sum_{\{k \in \Lambda^*,\, a < \|k\| < b\}} \omega_{\Lambda, \rho}^{B, \alpha}(N_k) \\ &= \partial_\alpha \lim_\Lambda \sup_{c \in \mathbb{C}} \tilde{p}_\Lambda^B(\beta, \mu_\Lambda^B(\theta, \rho), \alpha; c^\#) \end{aligned}\tag{2.18}$$

for $\alpha \in [-\delta, \delta]$. By explicit calculations in the right-hand side of (2.18) (see (A.4)–(A.6)) one obtains for $\alpha = 0$ equalities (2.15) and (2.16). ▮

Remark 2.3. Notice that the mean particle values $\omega_\Lambda^B(N_k) = \langle N_k \rangle_{H_\Lambda^B}(\beta, \mu)$ (and similar $\omega_{\Lambda, \rho}^B(N_k) = \langle N_k \rangle_{H_\Lambda^B}(\beta, \mu_\Lambda^B(\theta, \rho)))$ are defined on the discrete set Λ^* (1.3). Below we denote by $\{\omega_\Lambda^B(N_k)\}_{k \in \mathbb{R}^3}$ a continuous interpolation of these values from the set Λ^* to $\mathbb{R}^3$.

Now we are in position to prove the main statement of this section about the Bose–Einstein condensation showing up in the WIBG for densities $\rho > \rho_c^B(\theta)$.

Theorem 2.4. Let $\rho > \rho_c^B(\theta)$. Then we have that

(i)

$$\rho_0^B(\theta, 0) = \lim_\Lambda \omega_{\Lambda, \rho}^B\left(\frac{a_0^* a_0}{V}\right) = \begin{cases} |\hat{c}(\theta, 0)|^2, \theta < \theta_0(0) \\ 0, \theta > \theta_0(0) \end{cases} \tag{2.19}$$

(ii) for any $k \in \Lambda^*$, such that $\|k\| > 2\pi/L$,

$$\lim_\Lambda \omega_{\Lambda, \rho}^B\left(\frac{N_k}{V}\right) = 0 \tag{2.20}$$

(iii) for $\theta < \theta_0(0)$ and for all $k \in \Lambda^*$, such that $\|k\| > \delta > 0$

$$\lim_\Lambda \omega_{\Lambda, \rho}^B(N_k) = \left[\frac{f_k}{E_k}(e^{\beta E_k} - 1)^{-1} + \frac{h_k^2}{2E_k(f_k + E_k)}\right]_{c = \hat{c}(\theta, 0),\, \mu = 0} \tag{2.21}$$

whereas for $\theta > \theta_0(0)$

$$\lim_{\Lambda} \omega^{B}_{\Lambda,\rho}(N_k) = \frac{1}{e^{\beta\varepsilon_k} - 1} \tag{2.22}$$

(iv) the double limit

$$\rho^{B}_{0}(\theta) \equiv \lim_{\delta \to 0^+} \lim_{\Lambda} \frac{1}{V} \sum_{\{k \in \Lambda^*,\, 0 < \|k\| \leqslant \delta\}} \omega^{B}_{\Lambda,\rho}(N_k) = \rho - \rho^{B}_{c}(\theta) \tag{2.23}$$

which means that the WIBG manifests a conventional (generalized) Bose condensation $\tilde{\rho}^{B}_{0}(\theta) > 0$ in modes next to the zero-mode due to particle density saturation.

Proof. (i) Since by (2.9) we have

$$\lim_{\Lambda} \mu^{B}_{\Lambda}(\theta, \rho) = 0 \tag{2.24}$$

the thermodynamic limit (2.19) results from Theorem 4.4 and Corollary 4.8 of ref. 5, see (1.16) for $\mu = 0$.

(ii) Since $\|k\| < 2\pi/L$ and $\Lambda = L \times L \times L$ is a cube, which excludes a generalized Bose–Einstein condensation due to anisotropy (see ref. 12 or Appendix B), the thermodynamic limit (2.20) follows from $\mu^{B}_{\Lambda}(\theta, \rho) < \varepsilon_{\|k\| = 2\pi/L}$ (Lemma D.1) and the estimate (D.10) of Lemma D.2.

(iii) Let us consider $g_\theta(k)$ defined for $k \in \mathbb{R}^3$, $\|k\| > \delta > 0$ by

$$g_\theta(k) \equiv \lim_{\Lambda} \omega^{B}_{\Lambda,\rho}(N_k) \tag{2.25}$$

where the state $\omega^{B}_{\Lambda,\rho}(-)$ stands for $\omega^{B}_{\Lambda}(-)$ with $\mu = \mu^{B}_{\Lambda}(\theta, \rho)$, cf. (2.10). Notice that by Lemma D.2 and by the fact that (see Lemma D.1)

$$\mu^{B}_{\Lambda}(\theta, \rho) < \varepsilon_{\Lambda,1} < \inf_{k \neq 0} \varepsilon_k = \varepsilon_{\|k\| = 2\pi/L}$$

the thermodynamic limit (2.25) exists and it is informly bounded for $\|k\| > \delta > 0$. Moreover, for any interval $(a > \delta, b)$ we have

$$\lim_{\Lambda} \frac{1}{V} \sum_{\{k \in \Lambda^*,\, \|k\| \in (a,b)\}} \omega^{B}_{\Lambda,\rho}(N_k) = \frac{1}{(2\pi)^3} \int_{\|k\| > \delta} g_\theta(k)\, \chi_{(a,b)}(\|k\|)\, d^3k$$

where again $\chi_{(a,b)}(\|k\|)$ is the characteristic function of (a,b). Then Corollary 2.2 implies that

$$\frac{1}{(2\pi)^3}\int_{\|k\|>\delta} g_\theta(k)\,\chi_{(a,b)}(\|k\|)\,d^3k = \frac{1}{(2\pi)^3}\int_{\|k\|>\delta} f_\theta(k)\,\chi_{(a,b)}(\|k\|)\,d^3k \tag{2.26}$$

where $f_\theta(k)$ is a continuous function on $k\in\mathbb{R}^3$ defined by (2.15), (2.16), i.e.,

$$f_\theta(k)\equiv\frac{1}{(2\pi)^3}\left[\frac{f_k}{E_k}(e^{\beta E_k}-1)^{-1}+\frac{h_k^2}{2E_k(f_k+E_k)}\right]_{c=\hat{c}(\theta,0),\,\mu=0} \tag{2.27}$$

for $\theta<\theta_0(0)$ and

$$f_\theta(k)\equiv\frac{1}{(2\pi)^3}(e^{\beta\varepsilon_k}-1)^{-1} \tag{2.28}$$

for $\theta>\theta_0(0)$. Since the relation (2.26) is valid for any interval $(a>\delta, b)\subset\mathbb{R}$, one gets

$$g_\theta(k)=f_\theta(k),\qquad k\in\mathbb{R}^3,\quad \|k\|>\delta>0$$

By this and (2.25)–(2.28) we deduce (2.21) and (2.22).

(iv) Since the total density ρ is fixed, by definition (2.10) we have

$$\frac{1}{V}\sum_{\{k\in\Lambda^*,\,0<\|k\|\leqslant\delta\}}\omega^B_{\Lambda,\rho}(N_k)=\rho-\omega^B_{\Lambda,\rho}\left(\frac{a_0^*a_0}{V}\right)-\frac{1}{V}\sum_{\{k\in\Lambda^*,\,\|k\|>\delta\}}\omega^B_{\Lambda,\rho}(N_k) \tag{2.29}$$

By Corollary 2.2 for $a=\delta$ and $b\to+\infty$ we obtain for $\theta<\theta_0(0)$

$$\begin{aligned}\lim_\Lambda\frac{1}{V}&\sum_{\{k\in\Lambda^*,\,\|k\|>\delta\}}\omega^B_{\Lambda,\rho}(N_k)\\&=\frac{1}{(2\pi)^3}\int_{\|k\|>\delta}\left[\frac{f_k}{E_k}(e^{\beta E_k}-1)^{-1}+\frac{h_k^2}{2E_k(f_k+E_k)}\right]_{c=\hat{c}(\theta,0),\,\mu=0}d^3k\end{aligned} \tag{2.30}$$

and for $\theta>\theta_0(0)$

$$\lim_\Lambda\frac{1}{V}\sum_{\{k\in\Lambda^*,\,\|k\|>\delta\}}\omega^B_{\Lambda,\rho}(N_k)=\frac{1}{(2\pi)^3}\int_{\|k\|>\delta}(e^{\beta\varepsilon_k}-1)^{-1}\,d^3k \tag{2.31}$$

Now, from (2.3), (2.4), (2.19), (2.29)–(2.31) we deduce (2.23) by taking the limit $\delta\to0^+$. ∎

Therefore, according to (2.23) (and in a close similarity to ref. 11) for $\theta > \theta_0(0)$ and $\rho > \rho_c^B(\theta)$ the WIBG manifests only *one kind* of condensation, namely the *conventional* Bose–Einstein condensation which occurs in modes $k \neq 0$, whereas for $\theta < \theta_0(0)$ it manifests for $\rho > \rho_c^B(\theta)$ this kind of condensation at the *second* stage after the *nonconventional* Bose condensation $|\hat{c}(\theta, 0)|^2$, see (2.19). For classification of different condensations see Appendix B.

Remark 2.5. In domain: $\theta < \theta_0(0)$, $\rho > \rho_c^B(\theta)$, we have coexistence of these two kinds of condensations, namely:

— the nonconventional one, which starts when ρ becomes larger than $\rho_{\sup}^B(\theta)$, see (2.5) and Fig. 2 (b), and which reaches its maximal value $\rho_0^B(\theta, 0)$ for $\rho \geqslant \rho_c^B(\theta) > \rho_{\sup}^B(\theta)$;

— and the conventional Bose condensation $\tilde{\rho}_0^B(\theta)$ which appears when $\rho > \rho_c^B(\theta)$, see (2.23).

Since the Bose–Einstein condensation (2.23) occurs in modes $k \neq 0$, it should be classified as a *generalized* condensation. According to the van den Berg–Lewis–Pulè's classification (see refs. 12, 13, 14 and Appendix B), from (2.20) and (2.23) we can deduce only that the *generalized conventional* condensation in the WIBG can be either a condensation of type I in modes $\|k\| = 2\pi/L$, or a condensation of type III if modes $\|k\| \geqslant 2\pi/L$ are not macroscopically occupied (non-extensive condensation), or finally it can be a combination of the both.

Corollary 2.6. For $\rho > \rho_c^B(\theta)$ and periodic boundary conditions (1.3) the (generalized) conventional condensation (2.23) is of type I in the first $2d(=6)$ modes next to the zero-mode $k = 0$, i.e.

$$\tilde{\rho}_0^B(\theta) = \lim_{\Lambda} \frac{1}{V} \sum_{\{k \in \Lambda^*,\ \|k\| = 2\pi/L\}} \omega_{\Lambda,\rho}^B(a_k^* a_k) = \rho - \rho_c^B(\theta) \tag{2.32}$$

Proof. Since for $\delta > 0$

$$\begin{aligned}\frac{1}{V} &\sum_{\{k \in \Lambda^*,\ \|k\| = 2\pi/L\}} \omega_{\Lambda,\rho}^B(N_k) \\ &= \rho - \omega_{\Lambda,\rho}^B\left(\frac{a_0^* a_0}{V}\right) - \frac{1}{V} \sum_{\{k \in \Lambda^*,\ 2\pi/L < \|k\| < \delta\}} \omega_{\Lambda,\rho}^B(N_k) \\ &\quad - \frac{1}{V} \sum_{\{k \in \Lambda^*,\ \|k\| \geqslant \delta\}} \omega_{\Lambda,\rho}^B(N_k)\end{aligned}$$

by Lemma D.2 we get

$$\begin{aligned}
&\frac{1}{V} \sum_{\{k \in \Lambda^*, \, \|k\| = 2\pi/L\}} \omega^B_{\Lambda,\rho}(N_k) \\
&\quad \geqslant \rho - \frac{1}{V} \sum_{\{k \in \Lambda^*, \, 2\pi/L < \|k\| < \delta\}} \frac{1}{e^{B_k(\mu^B_\Lambda(\theta,\rho))} - 1} \\
&\quad - \omega^B_{\Lambda,\rho}\left(\frac{a_0^* a_0}{V}\right) \left[1 + \frac{\beta}{2V} \sum_{\{k \in \Lambda^*, \, 2\pi/L < \|k\| < \delta\}} \frac{v(k)}{1 - e^{-B_k(\mu^B_\Lambda(\theta,\rho))}} \right] \\
&\quad - \frac{1}{V} \sum_{\{k \in \Lambda^*, \, \|k\| \geqslant \delta\}} \omega^B_{\Lambda,\rho}(N_k)
\end{aligned} \tag{2.33}$$

with $B_k(\mu^B_\Lambda(\theta, \rho))$ defined by (D.11). Since by Lemma D.1 one has

$$\mu^B_\Lambda(\theta, \rho) < \varepsilon_{\Lambda,1} < \inf_{k \neq 0} \varepsilon_k = \varepsilon_{\|k\| = 2\pi/L}$$

from (2.3), (2.4) and (2.30) we deduce that

$$\lim_\Lambda \frac{1}{V} \sum_{\{k \in \Lambda^*, \, \|k\| = 2\pi/L\}} \omega^B_{\Lambda,\rho}(a_k^* a_k) \geqslant \rho - \rho^B_c(\theta) \tag{2.34}$$

by taking the limit $\delta \to 0^+$ in the right-hand side of (2.33) *after* the thermodynamic limit. Hence combining the inequality

$$\lim_\Lambda \frac{1}{V} \sum_{\{k \in \Lambda^*, \, \|k\| = 2\pi/L\}} \omega^B_{\Lambda,\rho}(N_k) \leqslant \lim_\Lambda \frac{1}{V} \sum_{\{k \in \Lambda^*, \, 0 < \|k\| < \delta\}} \omega^B_{\Lambda,\rho}(N_k)$$

with (2.23) and (2.34), we obtain (2.32). ∎

Therefore, for temperature θ and total particle density ρ as parameters, we obtain *three regimes* in thermodynamic behaviour of the WIBG when $\theta < \theta_0(0)$ (see Figs. 1 and 2):

(i) for $\rho \leqslant \rho^B_{\inf}(\theta)$, there is no condensation;

(ii) for $\rho^B_{\sup}(\theta) \leqslant \rho \leqslant \rho^B_c(\theta)$, there is a *nonconventional* condensation (1.16) in the mode $k = 0$ due to nondiagonal interaction in the Bogoliubov Hamiltonian, see Fig. 1;

(iii) for $\rho^B_c(\theta) \leqslant \rho$, there is a *second* kind of condensation: the *conventional* type I Bose–Einstein condensation which occurs *after* the *nonconventional* one; it appears due to the standard mechanism of the total particle density saturation (Corollary 2.6).

When $\theta > \theta_0(0)$, there are only *two* types of thermodynamic behaviour: they correspond to $\rho \leqslant \rho_c^B(\theta)$ with no condensation and to $\rho_c^B(\theta) < \rho$ with a *conventional* condensation as in (iii). Hence, for $\theta > \theta_0(0)$ the condensation in the WIBG coincides with type I generalized Bose–Einstein condensation in the PBG with *excluded* mode $k=0$, see Theorem 2.4 (iii) and ref. 15.

3. APPROXIMATING STATE FOR THE WIBG

By definition of the coherent states (A.2) in the zero-mode of the boson Fock space $\mathscr{F}_\Lambda$ and by (1.8) one gets

$$p_\Lambda^B(\beta,\mu) = \frac{1}{\beta V} \ln \frac{1}{2\pi} \int_{\mathbb{C}} dc\, d\bar{c}\, Tr_{\mathscr{F}'_\Lambda} W_\Lambda(c^\#) \equiv \frac{1}{\beta V} \ln \frac{1}{2\pi} \int_{\mathbb{C}} dc\, d\bar{c}\, e^{\beta V p_\Lambda^B(\beta,\mu;\, c^\#)}$$

Here $W_\Lambda(c^\#)$ results from the *Bogoliubov approximation* (see Appendix A) for the statistical operator W_Λ

$$W_\Lambda = e^{-\beta(H_\Lambda^B - \mu N_\Lambda)}$$

and $p_\Lambda^B(\beta,\mu;\, c^\#)$ is the pressure defined by the partial trace:

$$p_\Lambda^B(\beta,\mu;\, c^\#) \equiv \frac{1}{\beta V} \ln Tr_{\mathscr{F}'_\Lambda} W_\Lambda(c^\#)$$

On the other hand, one can also define the pressure $\hat{p}_\Lambda^B(\beta,\mu)$ corresponding to the *Bogoliubov approximation* in Hamiltonian H_Λ^B (cf. (A.3), (A.4)) by

$$\begin{aligned}\hat{p}_\Lambda^B(\beta,\mu) &\equiv \frac{1}{\beta V} \ln \frac{1}{2\pi} \int_{\mathbb{C}} dc\, d\bar{c}\, Tr_{\mathscr{F}'_\Lambda} e^{-\beta H_\Lambda^B(c^\#,\mu)} \\ &= \frac{1}{\beta V} \ln \frac{1}{2\pi} \int_{\mathbb{C}} dc\, d\bar{c}\, e^{\beta V \tilde{p}_\Lambda^B(\beta,\mu;\, c^\#)}\end{aligned} \tag{3.1}$$

The difference is that $p_\Lambda^B(\beta,\mu)$ results from the integration over the complex plane of the trace of Bogoliubov approximation for the statistical operator whereas for the pressure $\hat{p}_\Lambda^B(\beta,\mu)$ the Bogoliubov approximation is done directly in the Bogoliubov Hamiltonian.

The aim of this section is to construct and to study the approximating state $\hat{\omega}_\Lambda^B(-)$ corresponding to the pressure $\hat{p}_\Lambda^B(\beta,\mu)$ (3.1) in order to understand how it is related to the Gibbs state $\omega_\Lambda^B(-)$ (1.17).

First, by standard large deviation arguments (see e.g., refs. 16, 17), one can prove that for any $\mu \leqslant 0$

$$\hat{p}^B(\beta,\mu) \equiv \lim_{\Lambda} \hat{p}^B_\Lambda(\beta,\mu) = \sup_{c\in\mathbb{C}} \tilde{p}^B(\beta,\mu;c^{\#}) = p^B(\beta,\mu) \tag{3.2}$$

see Lemma E.1 in Appendix E. This result à la Ginibre[7] is known for the superstable Bose gas. Next, the grand-canonical Gibbs state for the WIBG (1.17)

$$\omega^B_\Lambda(-) = e^{\beta V p^B_\Lambda(\beta,\mu)} Tr_{\mathcal{F}_\Lambda}[(-)\, e^{-\beta(H^B_\Lambda - \mu N_\Lambda)}]$$

is defined on the operator C^*-algebra $\mathscr{A}_\Lambda$ living on the boson Fock space $\mathscr{F}_\Lambda$. The tensor structure $\mathscr{F}_\Lambda \approx \mathscr{F}_{0\Lambda} \otimes \mathscr{F}'_\Lambda$ (A.1) implies that $\mathscr{A}_\Lambda \approx \mathscr{A}_{0\Lambda} \otimes \mathscr{A}'_\Lambda$ where $\mathscr{A}_{0\Lambda}$ and $\mathscr{A}'_\Lambda$ are C^*-algebras on the boson Fock spaces $\mathscr{F}_{0\Lambda}$ and $\mathscr{F}'_\Lambda$ respectively. Therefore, we are looking for a trial approximating state $\hat{\omega}^B_\Lambda$ as a convex combination of product states on $\mathscr{A}_{0\Lambda} \otimes \mathscr{A}'_\Lambda$:

$$\hat{\omega}^B_\Lambda = \int_{\mathbb{D}_\nu} d\nu(\lambda)\, \eta_{\Lambda,\lambda} \otimes \omega_{\Lambda,\lambda} \tag{3.3}$$

Here $\eta_{\Lambda,\lambda}$, $\omega_{\Lambda,\lambda}$, $\lambda \in \mathbb{D}_\nu$, are two families of states on $\mathscr{A}_{0\Lambda}$ and on $\mathscr{A}'_\Lambda$ respectively and the probability measure $d\nu(\lambda)$ has support $\mathbb{D}_\nu$.

Remark 3.1. The form of the approximating state $\hat{\omega}^B_\Lambda$ (3.3), defined here as an "ansatz," can be understood with a help of ref. 18 and refs. 19, 20. Indeed, Størmer[18] proved a theorem on the integral decomposition of symmetric states of an infinite tensor product of C^*-algebra $\bigotimes_{j=1}^{+\infty} \mathscr{B}_j$. The Fannes–Lewis–Verbeure theorem[19] extends this result to symmetric states on a composite algebra $\mathscr{A}_0 \otimes (\bigotimes_{j=1}^{+\infty} \mathscr{B}_j)$. In our case the state ω^B_Λ is defined on infinite tensor product for the lattice Λ^* (1.3). Since

$$\omega^B_\Lambda(N_{k_1}) \neq \omega^B_\Lambda(N_{k_2}), \qquad k_1 \neq k_2$$

this state is not symmetric. Hence one has to consider (3.3) as an ansatz.

Definition 3.2. We define the approximating state $\hat{\omega}^B_\Lambda$ corresponding to the pressure $\hat{p}^B_\Lambda(\beta,\mu)$ (3.1) by

$$\hat{\omega}^B_\Lambda(X = X_0 \otimes X_1) \equiv \int_{\mathbb{C}} \mathbb{K}^\mu_\Lambda(dc\, d\bar{c})\, \eta_{\Lambda,c}(X_0) \otimes \omega_{\Lambda,c}(X_1) \tag{3.4}$$

where $X_0 \in \mathscr{A}_{0\Lambda}$, $X_1 \in \mathscr{A}'_\Lambda$, and $\mathbb{K}^\mu_\Lambda(dc\, d\bar{c})$ is a probability measure on the complex plane $\mathbb{C}$ defined by

$$\mathbb{K}^\mu_\Lambda(dc\, d\bar{c}) \equiv \frac{e^{-\beta V \hat{p}^B_\Lambda(\beta,\mu)}}{2\pi}\, dc\, d\bar{c}\; Tr_{\mathscr{F}'_\Lambda} e^{-\beta H^B_\Lambda(c^\#,\mu)} \tag{3.5}$$

Here $\eta_{\Lambda,c}$ is a (quasi)-free state[21, 22] defined by

$$\begin{aligned} \eta_{\Lambda,c}(a_0) &= c\sqrt{V} \\ \eta_{\Lambda,c}(a_0^* a_0) &= \eta_{\Lambda,c}(a_0^*)\, \eta_{\lambda,c}(a_0) = |c|^2\, V \\ \eta_{\Lambda,c}(a_0^2) &= \eta_{\Lambda,c}(a_0)\, \eta_{\Lambda,c}(a_0) = c^2 V \\ \eta_{\Lambda,c}(a_0^{*2}) &= \eta_{\Lambda,c}(a_0^*)\, \eta_{\Lambda,c}(a_0^*) = \bar{c}^2 V \end{aligned} \tag{3.6}$$

and $\omega_{\Lambda,c}$ is a $\mathbb{K}^\mu_\Lambda$-measurable state on $\mathscr{A}'_\Lambda$ defined by

$$\omega_{\Lambda,c}(-) \equiv \frac{Tr_{\mathscr{F}'_\Lambda}[(-)\, e^{-\beta H^B_\Lambda(c^\#,\mu)}]}{Tr_{\mathscr{F}'_\Lambda} e^{-\beta H^B_\Lambda(c^\#,\mu)}} \tag{3.7}$$

Theorem 3.3. For any $\mu < 0$ such that $\mu \neq \mu_0(\theta)$, we have

$$\lim_\Lambda \hat{\omega}^B_\Lambda\left(\frac{a_0^* a_0}{V}\right) = \lim_\Lambda \omega^B_\Lambda\left(\frac{a_0^* a_0}{V}\right)$$

and

$$\lim_\Lambda \hat{\omega}^B_\Lambda\left(\frac{N_\Lambda}{V}\right) = \lim_\Lambda \omega^B_\Lambda\left(\frac{N_\Lambda}{V}\right) = \rho^B(\theta,\mu)$$

Proof. Since

$$\frac{N_\Lambda}{V} = \frac{a_0^* a_0}{V} \otimes \mathbb{I}' + \mathbb{I}_0 \otimes \frac{N'_\Lambda}{V}$$

by explicit calculations one gets from (3.4)–(3.7) that

$$\begin{aligned} \hat{\omega}^B_\Lambda\left(\frac{N_\Lambda}{V}\right) &= \int_{\mathbb{C}} \mathbb{K}^\mu_\Lambda(dc\, d\bar{c}) \left[\eta_{\Lambda,c}\left(\frac{a_0^* a_0}{V}\right) \otimes \omega_{\Lambda,c}(\mathbb{I}') + \eta_{\Lambda,c}(\mathbb{I}_0) \otimes \omega_{\Lambda,c}\left(\frac{N'_\Lambda}{V}\right)\right] \\ &= \int_{\mathbb{C}} \mathbb{K}^\mu_\Lambda(dc\, d\bar{c})\, |c|^2 + \frac{1}{V} \sum_{k\in\Lambda^*, k\neq 0} \int_{\mathbb{C}} \mathbb{K}^\mu_\Lambda(dc\, d\bar{c}) \\ &\quad \times \left[\frac{f_k}{E_k}(e^{\beta E_k} - 1)^{-1} + \frac{h_k^2}{2E_k(f_k + E_k)}\right] \end{aligned} \tag{3.8}$$

for $\mu < 0$. Then by (2.1) and Lemma E.2 (ii) of Appendix E one obtains

$$\lim_{\Lambda} \hat{\omega}_{\Lambda}^{B}\left(\frac{N_{\Lambda}}{V}\right) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} (e^{\beta(\varepsilon_k - \mu)} - 1)^{-1} \, d^3k = \rho^B(\theta, \mu)$$

for $\mu < \mu_0(\theta)$, whereas by (2.2) and Lemma E.2 (i) we get

$$\lim_{\Lambda} \hat{\omega}_{\Lambda}^{B}\left(\frac{N_{\Lambda}}{V}\right) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \left[\frac{f_k}{E_k}(e^{\beta E_k} - 1)^{-1} + \frac{h_k^2}{2E_k(f_k + E_k)}\right]_{c = \hat{c}(\theta, \mu)} d^3k \\ + |\hat{c}(\theta, \mu)|^2 = \rho^B(\theta, \mu) \tag{3.9}$$

for $\mu_0(\theta) < \mu < 0$. ∎

Remark 3.4. From (A.5), (E.4) and estimate (E.8), we find that $\hat{p}_{\Lambda}^{B}(\beta, \mu)$ (3.1) is bounded only for $\mu < \hat{\varepsilon}_{\Lambda, 1} \equiv \inf_{k \neq 0} \varepsilon_k$. Since by convexity with respect to μ one has for $0 < \mu < \hat{\varepsilon}_{\Lambda, 1}$:

$$\frac{\hat{p}_{\Lambda}^{B}(\beta, \mu) - \hat{p}_{\Lambda}^{B}(\beta, 0)}{\mu} \leqslant \partial_{\mu} \hat{p}_{\Lambda}^{B}(\beta, \mu) = \hat{\omega}_{\Lambda}^{B}\left(\frac{N_{\Lambda}}{V}\right)$$

where the last equality is due to definitions (3.1), (3.4) and (3.8), then

$$\lim_{\mu \to \hat{\varepsilon}_{\Lambda, 1}} \hat{\omega}_{\Lambda}^{B}\left(\frac{N_{\Lambda}}{V}\right) = +\infty \tag{3.10}$$

Therefore, if $\rho \geqslant \rho_c^B(\theta)$ there is a unique $\hat{\mu}_{\Lambda}^{B}(\theta, \rho) < \hat{\varepsilon}_{\Lambda, 1}$ such that

$$\hat{\omega}_{\Lambda}^{B}\left(\frac{N_{\Lambda}}{V}\right) = \rho$$

and

$$\lim_{\Lambda} \hat{\mu}_{\Lambda}^{B}(\theta, \rho \geqslant \rho_c^B(\theta)) = 0 \tag{3.11}$$

Below we put (cf. (2.10))

$$\hat{\omega}_{\Lambda, \rho}^{B}(-) \equiv \hat{\omega}_{\Lambda}^{B}(-)|_{\mu = \hat{\mu}_{\Lambda}^{B}(\theta, \rho)} \tag{3.12}$$

Theorem 3.5. If $\rho \geqslant \rho_c^B(\theta)$, where $\rho_c^B(\theta)$ is the critical density defined by (2.3), (2.4), then we have

$$\hat{\rho}_0^B(\theta) \equiv \lim_{\Lambda} \frac{1}{V} \sum_{\{k \in \Lambda^*,\ \|k\| = 2\pi/L\}} \hat{\omega}_{\Lambda,\rho}^B(a_k^* a_k) = \rho - \rho_c^B(\theta) \tag{3.13}$$

Proof. Notice that

$$\sum_{\{k \in \Lambda^* :\ \|k\| = 2\pi/L\}} \hat{\omega}_{\Lambda,\rho}^B(N_k)$$
$$= \hat{\omega}_{\Lambda,\rho}^B(N_\Lambda) - \hat{\omega}_{\Lambda,\rho}^B(a_0^* a_0) - \sum_{\{k \in \Lambda^*,\ \|k\| > 2\pi/L\}} \hat{\omega}_{\Lambda,\rho}^B(N_k) \tag{3.14}$$

and

$$\sum_{\{k \in \Lambda^*,\ \|k\| > 2\pi/L\}} \hat{\omega}_{\Lambda,\rho}^B(N_k)$$
$$= \sum_{\{k \in \Lambda^*,\ \|k\| \geqslant \delta\}} \hat{\omega}_{\Lambda,\rho}^B(N_k) + \sum_{\{k \in \Lambda^*,\ 2\pi/L < \|k\| < \delta\}} \hat{\omega}_{\Lambda,\rho}^B(N_k) \tag{3.15}$$

where $\delta < 0$. Since by (3.4)–(3.7)

$$\sum_{\{k \in \Lambda^*,\ \|k\| \geqslant \delta\}} \hat{\omega}_{\Lambda,\rho}^B(N_k)$$
$$= \sum_{\{k \in \Lambda^*,\ \|k\| \geqslant \delta\}} \int_{\mathbb{C}} \mathbb{K}_{\Lambda}^{\hat{\mu}_\Lambda^B(\theta,\rho)}(dc\, d\bar{c}) \left[\frac{f_k}{E_k} (e^{\beta E_k} - 1)^{-1} + \frac{h_k^2}{2E_k(f_k + E_k)} \right]$$

then by (3.11) and Lemma E.2 we obtain that for $\theta < \theta_0(0)$

$$\lim_{\delta \to 0^+} \lim_{\Lambda} \frac{1}{V} \sum_{\{k \in \Lambda^*,\ \|k\| \geqslant \delta\}} \hat{\omega}_{\Lambda,\rho}^B(N_k)$$
$$= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \left[\frac{f_k}{E_k} (e^{\beta E_k} - 1)^{-1} + \frac{h_k^2}{2E_k(f_k + E_k)} \right]_{c = \hat{c}(\theta,0),\ \mu = 0} d^3k$$

whereas for $\theta > \theta_0(0)$

$$\lim_{\delta \to 0^+} \lim_{\Lambda} \frac{1}{V} \sum_{\{k \in \Lambda^*,\ \|k\| \geqslant \delta\}} \hat{\omega}_{\Lambda,\rho}^B(N_k) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} (e^{\beta \varepsilon_k} - 1)^{-1}\, d^3k$$

Hence, from (1.16), (2.3) and (2.4) we get

$$\lim_{\delta\to 0^+}\lim_{\Lambda}\frac{1}{V}\sum_{\{k\in\Lambda^*,\ \|k\|\geqslant\delta\}}\hat{\omega}^B_{\Lambda,\rho}(N_k)=\rho^B_c(\theta)-\rho^B_0(\theta,0) \tag{3.16}$$

for any $\theta\geqslant 0$. The second term in the right-hand side of (3.15) can be rewritten as

$$\begin{aligned}
&\sum_{\{k\in\Lambda^*,\ 2\pi/L<\|k\|<\delta\}}\hat{\omega}^B_{\Lambda,\rho}(N_k)\\
&\quad=\sum_{\{k\in\Lambda^*,\ 2\pi/L<\|k\|<\delta\}}\int_{\mathbb{C}}\mathbb{K}_\Lambda^{\hat{\mu}^B_\Lambda(\theta,\rho)}(dc\,d\bar{c})\\
&\qquad\times\left[\frac{f_k}{E_k}(e^{\beta E_k}-1)^{-1}+\frac{h_k^2}{2E_k(f_k+E_k)}\right]\\
&\quad=\sum_{\{k\in\Lambda^*,\ 2\pi/L<\|k\|<\delta\}}\int_0^{\varepsilon}\widetilde{\mathbb{K}}_\Lambda^{\hat{\mu}^B_\Lambda(\theta,\rho)}(dx)\,G_k(x,\mu)\\
&\qquad+\sum_{\{k\in\Lambda^*,\ 2\pi/L<\|k\|<\delta\}}\int_{\varepsilon}^{+\infty}\widetilde{\mathbb{K}}_\Lambda^{\hat{\mu}^B_\Lambda(\theta,\rho)}(dx)\,G_k(x,\mu)
\end{aligned} \tag{3.17}$$

for some $\varepsilon>0$, with

$$\widetilde{\mathbb{K}}^{\mu}_\Lambda(dx)=e^{\beta V[\tilde{p}^B_\Lambda(\mu,x)-\hat{p}^B_\Lambda(\beta,\mu)]}dx \tag{3.18}$$

and

$$G_k(x,\mu)=\frac{f_k}{E_k}(e^{\beta E_k}-1)^{-1}+\frac{h_k^2}{2E_k(f_k+E_k)} \tag{3.19}$$

If $\varepsilon<|\hat{c}(\theta,0)|^2$, then by Lemma E.2 one gets

$$\begin{aligned}
&\lim_{\Lambda}\frac{1}{V}\sum_{\{k\in\Lambda^*:\ 2\pi/L<\|k\|<\delta\}}\int_{\varepsilon}^{+\infty}\widetilde{\mathbb{K}}_\Lambda^{\hat{\mu}^B_\Lambda(\theta,\rho)}(dx)\,G_k(x,\hat{\mu}^B_\Lambda(\theta,\rho))\\
&\quad=\frac{1}{(2\pi)^3}\int_{\|k\|<\delta}G_k(\hat{x}(\theta,0),0)\,d^3k
\end{aligned}$$

where $\hat{x}(\theta,0)=|\hat{c}(\theta,0)|^2$ and thus by (3.19) we have

$$\lim_{\delta\to 0^+}\lim_{\Lambda}\frac{1}{V}\sum_{\{k\in\Lambda^*,\ 2\pi/L<\|k\|<\delta\}}\int_{\varepsilon}^{+\infty}\widetilde{\mathbb{K}}_\Lambda^{\hat{\mu}^B_\Lambda(\theta,\rho)}(dx)\,G_k(x,\hat{\mu}^B_\Lambda(\theta,\rho))=0 \tag{3.20}$$

By continuity of $G_k(x, \mu)$, one obtains that for $2\pi/L < \|k\| < \delta$

$$\int_0^\varepsilon \widetilde{\mathbb{K}}_\Lambda^{\hat{\mu}_\Lambda^B(\theta,\rho)}(dx)\, G_k(x, \hat{\mu}_\Lambda^B(\theta,\rho)) \leqslant [\sup_{x\in[0,\varepsilon]} G_k(x, \hat{\varepsilon}_{\lambda,1})] \int_0^\varepsilon \widetilde{\mathbb{K}}_\Lambda^{\hat{\mu}_\Lambda^B(\theta,\rho)}(dx) \tag{3.21}$$

Since by (3.19) the function

$$\widetilde{G}(k) \equiv \sup_{x\in[0,\varepsilon]} G_k(x, \hat{\varepsilon}_{\Lambda,1})$$

is integrable over k, then combining the Lebesgue dominate convergence theorem with Lemma E.2 we deduce from (3.21) that

$$\lim_{\delta\to 0^+} \lim_\Lambda \sum_{\{k\in\Lambda^*,\, 2\pi/L<\|k\|<\delta\}} \int_0^\varepsilon \widetilde{\mathbb{K}}_\Lambda^{\hat{\mu}_\Lambda^B(\theta,\rho)}(dx)\, G_k(x, \hat{\mu}_\Lambda^B(\theta,\rho)) = 0 \tag{3.22}$$

Therefore, by the representation (3.17) and by the thermodynamic limits (3.20) and (3.22) one gets

$$\lim_{\delta\to 0^+} \lim_\Lambda \frac{1}{V} \sum_{\{k\in\Lambda^*,\, 2\pi/L<\|k\|<\delta\}} \hat{\omega}_{\Lambda,\rho}^B(N_k) = 0 \tag{3.23}$$

which together with (3.15) and (3.16) implies

$$\lim_\Lambda \frac{1}{V} \sum_{\{k\in\Lambda^*,\, \|k\|>2\pi/L\}} \hat{\omega}_{\Lambda,\rho}^B(N_k) = \rho_c^B(\theta) - \rho_0^B(\theta, 0) \tag{3.24}$$

Since by (1.16), (3.4)–(3.7) and Lemma E.2 one gets

$$\lim_\Lambda \hat{\omega}_{\Lambda,\rho}^B\left(\frac{a_0^* a_0}{V}\right) = \rho_0^B(\theta, 0)$$

from (3.14) and (3.24) we finally obtain (3.13). ∎

Summarizing Theorems 3.3 and 3.5 we conclude that for a fixed total particle density ρ the *approximating* state $\hat{\omega}_\Lambda^B$ gives for the density $\hat{\rho}_0^B(\theta)$ of condensate the same expression (3.13) as the state ω_Λ^B, cf. (2.32). The remarks (i)–(iii) formulated at the end of Section 2 are also valid in this case.

A new observation concerns the properties of the approximating state $\hat{\omega}_\Lambda^B$ for

$$\rho_{\inf}^B(\theta) < \rho < \rho_{\sup}^B(\theta) \leqslant \rho_c^B(\theta)$$

see Fig. 2 and (2.5), i.e., for the line of first-order phase transitions

$$\lim_{\Lambda} \hat{\mu}_{\Lambda}^{B}(\theta, \rho) = \mu_0(\theta)$$

Indeed, Lemma E.3 suggests a possibility of a mixture of two states: a state without condensation and a state with condensation in the mode $k=0$.

Theorem 3.6. Let $\gamma \in \mathbb{R}$. Then for $\mu = \mu_{\Lambda}$, where

$$\mu_{\Lambda} \equiv \mu_0(\theta) + \frac{\gamma}{\beta V} \tag{3.25}$$

one gets

$$\lim_{\Lambda} \hat{\omega}_{\Lambda}^{B}\left(\frac{a_0^* a_0}{V}\right) = (1-\lambda_\gamma)\, |\hat{c}(\theta, \mu_0(\theta))|^2 \tag{3.26}$$

and

$$\lim_{\Lambda} \hat{\omega}_{\Lambda}^{B}\left(\frac{N_{\Lambda}}{V}\right) = \lambda_\gamma \rho_{\inf}^{B}(\theta) + (1-\lambda_\gamma)\, \rho_{\sup}^{B}(\theta) \tag{3.27}$$

with

$$0 \leqslant \lambda_\gamma \equiv \frac{e^{\gamma \rho_{\inf}^{B}(\theta)}}{e^{\gamma \rho_{\inf}^{B}(\theta)} + e^{\gamma \rho_{\sup}^{B}(\theta)}} \leqslant 1 \tag{3.28}$$

see (E.17).

Proof. By virtue of (3.8), the proof of this theorem follows directly from Lemma E.3. ▌

In particular, for the case $\gamma = \gamma_{\Lambda} = \pm V^{\eta}$, $\eta < 1$, we get from (3.28):

$$\lim_{\gamma_{\Lambda} \to +\infty} \lambda_{\gamma_{\Lambda}} = 0$$

$$\lim_{\gamma_{\Lambda} \to -\infty} \lambda_{\gamma_{\Lambda}} = 1$$

Therefore, we can scan the whole interval $0 \leqslant \lambda_\gamma \leqslant 1$ with $\gamma \in \mathbb{R}$. The mixture of the two states in (3.26), (3.27) is in fact a direct consequence of the non-convexity of the trial pressure $\tilde{p}_{\Lambda}^{B}(\mu, x)$, see (E.5).

4. CONCLUDING REMARKS

In other papers[4, 5] we discussed the existence of a *nonconventional* condensation of bosons for $k=0$ for negative μ and $\theta<\theta_0(0)$ in the WIBG, see Fig. 1. The physical reason of this phenomenon is an *effective attraction* between bosons in the mode $k=0$:[6]

$$-\left\{\frac{1}{V^2}\sum_{k\in\Lambda^*,\,k\neq 0}\frac{[v(k)]^2}{4\varepsilon_k}\right\} a_0^{*2}a_0^2 \tag{4.1}$$

which has to dominate the direct repulsion in (1.6):

$$\frac{v(0)}{2V}\,a_0^{*2}a_0^2$$

This is formalized in the condition (C) (1.12).

In the present paper we show that conventional Bose–Einstein condensation is also possible in the WIBG. It is a *generalized* Bose–Einstein condensation of *type I* in modes $\|k\|=2\pi/L$. This second kind of condensation appears when $\rho\geqslant\rho_c^B(\theta)$ in accordance with a standard mechanism of the particle density saturation, which is well-known for the PBG, see Corollary 2.6.

Therefore, combining refs. 4, 5 with results of Section 2 for $\theta<\theta_0(0)$ (see Figs. 1 and 2) we obtain three types of thermodynamic behaviour for the WIBG ($d=3$):

(i) for $\rho\leqslant\rho_{\inf}^B(\theta)$, there is no condensation;

(ii) for $\rho_{\sup}^B(\theta)\leqslant\rho\leqslant\rho_c^B(\theta)$, a *nonconventional* condensation (1.16) appears in the mode $k=0$, see Fig. 1;

(iii) for $\rho_c^B(\theta)\leqslant\rho$, the WIBG manifests a *conventional* Bose–Einstein condensation of type I (Corollary 2.6) simultaneously with *nonconventional* one. Therefore, two kinds of condensations coexist.

Notice that for $\theta>\theta_0(0)$ the thermodynamic behaviour of the WIBG coincides with that of the PBG with excluded mode $k=0$. Therefore, it manifests only the Bose–Einstein condensation of type I, see (iii) in Theorem 2.4 and ref. 15.

For $\theta<\theta_0(0)$, the thermodynamic behaviour of the WIBG is related to the two recent models[11] defined respectively by Hamiltonians

$$H_\Lambda^0\equiv T_\Lambda+U_\Lambda^0 \tag{4.2}$$

and

$$H^1_\Lambda = H^0_\Lambda + U_\Lambda \tag{4.3}$$

where

$$T_\Lambda = \sum_{k \in \Lambda^* \setminus \{0\}} \varepsilon_k a_k^* a_k, \qquad \varepsilon_{k \neq 0} = \hbar^2 k^2 / 2m$$

$$U^0_\Lambda = \varepsilon_0 a_0^* a_0 + \frac{g_0}{V} a_0^* a_0^* a_0 a_0, \qquad \varepsilon_0 \in \mathbb{R}^1, \quad g_0 > 0 \tag{4.4}$$

$$U_\Lambda = \frac{1}{V} \sum_{k \in \Lambda^*,\, k \neq 0} g_k(V)\, a_k^* a_k^* a_k a_k$$

with $0 < g_k(V) \leqslant \gamma_k V^{\alpha_k}$ for $k \in \Lambda^* \setminus \{0\}$, $\alpha_k \leqslant \alpha_+ < 1$ and $0 < \gamma_k \leqslant \gamma_+$. Notice that in these models, $\varepsilon_0 \in \mathbb{R}^1$ is not equal to $\varepsilon_{\|k\|=0} = 0$. The paper[11] shows a possibility of coexistence of two kinds of Bose condensations in the both models (4.2) and (4.3). The behaviour of the WIBG for $\theta < \theta_0(0)$ is closer to the model (4.2) than to (4.3) in the sense that the Bose gas (4.2) manifests the same three types of thermodynamic behaviour (i)–(iii) as above. The difference is in the absence of the limiting temperature $\theta_0(0)$ and of discontinuity of the condensate and the total particle density as functions of μ. The peculiarity of the model (4.3) is that under conditions $g_{k \neq 0}(V) \geqslant g_- > 0$ or $\inf_{\|k\| < \delta_0,\, V} g_k(V) > 0$ in a band $\delta_0 > 0$, the repulsion U_Λ (4.4) spreads out the *conventional* Bose–Einstein condensation (originally of type I in modes $\|k\| = 2\pi/L$) into Bose–Einstein condensation of type III (cf. ref. 10, 11). Notice that the conventional Bose condensation persists in the model (4.3) even if for $k \in \Lambda^* \setminus \{0\}$ $g_k(V) = \gamma_k V^{\alpha_k} \xrightarrow[V \to +\infty]{} +\infty$ ($\alpha_k \leqslant \alpha_+ < 1$) which is similar to the WIBG. There in the effective two-bosons *repulsion* for k, $q \neq 0$

$$g_{\Lambda, kq} a_k^* a_{-k}^* a_{-q} a_q$$

the "form-factor" $g_{\Lambda, kq} > 0$ diverges with volume as $V^{2/3}$, see ref. 6. However, an important difference is that this *effective* interaction (which is due to non-diagonal term (1.7)) is not able to spread out the Bose–Einstein condensation into the type III as in the model (4.3).

Concerning the thermodynamic behaviour of the WIBG for densities

$$\rho^B_{\text{inf}}(\theta) < \rho < \rho^B_{\text{inf}}(\theta)$$

Theorem 3.6 suggests a possibility of a mixture of the two pure approximating states $\hat{\omega}^B(-)$ corresponding, respectively, to the state without and with condensation in mode $k=0$. This is similar to what is known for the mean-field Bose gas models.[23, 24] We conjecture the same behaviour of the state $\lim \omega_{\Lambda}^{B}(-)$.

Comments concerning the dependence of the condensate on dimension merits a separate remark.

Remark 4.1. By virtue of ref. 5 the nonconventional condensation $\rho_0^B(\theta,\mu)=|\hat{c}(\theta,\mu)|^2$ exists and it is uniformly bounded in D. For low dimensions $d=1,2$, it is true for potentials satisfying (A), (B), and for $d=3,4,...$, if in addition, the condition (C) is satisfied, see (1.12). By (2.3) and (2.4) this implies that for $d=1,2$ there is no Bose–Einstein condensation for $\theta>\theta_0$ ($\mu=0$) (since $\rho_c^B(\theta)=+\infty$ (2.3)), whereas it coexists with nonconventional condensation for $\theta<\theta_0$ ($\mu=0$), when $\rho>\rho_c^B(\theta)<+\infty$ (2.4).

It makes a difference with models (4.2) and (4.3):[11] there for $d=1,2$ one has no conventional condensation for all $\theta\geqslant 0$. This difference comes from the fact that in the WIBG the nonconventional condensation $\hat{c}(\theta,\mu)$ ensures a convergence of the integral (2.4) for $d=1,2$, whereas in models (4.2) and (4.3) not.

Notice that one of the possibility to correct the instability of the WIBG for $\mu>0$ would be to add to H_{Λ}^{B} (1.4) the "forward-scattering" repulsive interaction between particles next to the mode $k=0$:

$$H_{\Lambda}=H_{\Lambda}^{B}+\frac{v(0)}{2V}\sum_{k,\,q\in\Lambda^{*}\backslash\{0\}} a_k^{*}a_q^{*}a_q a_k \tag{4.5}$$

In paper [3] the superstable Hamiltonian (4.5) was proposed to extract the Landau gapless spectrum by doing the *Bogoliubov approximation* only in the operator $H_{\Lambda}-v(0)\,a_0^{*2}a_0^2/2V$. In fact the problem of thermodynamics and of gapless spectrum in the stabilized WIBG models is rather delicate, see discussions in refs. 3, 25, 26. The reason is that the interaction in the WIBG is in fact of a long-range, which implies the appearance of the gap when one has a nonconventional condensation in the zero-mode, see ref. 5. We return to the model (4.5) elsewhere.

APPENDIX A. THE BOGOLIUBOV APPROXIMATION

A.1. Definition[7]

Let a system of identical bosons be enclosed in a cubic box Λ of the volume $V=|\Lambda|$ with periodic boundary conditions on $\partial\Lambda$. Let $\psi_0(x)=1/\sqrt{V}$ be the one-particle constant function corresponding to the mode

$k = 0$. We denote by $\mathscr{F}_{0\Lambda}$ the boson Fock spaces constructed on the one-dimensional Hilbert space $\mathscr{H}_{0\Lambda}$ spanned by ψ_0. If $\mathscr{F}'_\Lambda$ is the Fock space constructed on the orthogonal complement $\mathscr{H}^{\perp}_{0\Lambda}$, then the Fock space $\mathscr{F}_\Lambda$ of the system (i.e., constructed on the $L^2(\Lambda) = \mathscr{H}_{0\Lambda} \oplus \mathscr{H}^{\perp}_{0\Lambda}$) is naturally isomorphic (cf. 27, Ch. II.4) to the tensor product $\mathscr{F}_{0\Lambda} \otimes \mathscr{F}'_\Lambda$:

$$\mathscr{F}_\Lambda \approx \mathscr{F}_{0\Lambda} \otimes \mathscr{F}'_\Lambda \tag{A.1}$$

For any complex $c \in \mathbb{C}$, we can define in $\mathscr{F}_{0\Lambda}$ a coherent vector

$$\psi_{0\Lambda}(c) = e^{-V|c|^2/2} \sum_{k=0}^{\infty} \frac{1}{k!} (\sqrt{V}\, c)^k\, (a_0^*)^k\, \Omega_0 \tag{A.2}$$

where Ω_0 is the vacuum of $\mathscr{F}_\Lambda$ and therefore $a_0 \psi_{0\Lambda}(c) = c\sqrt{V}\, \psi_{0\Lambda}(c)$. Using this concept, Ginibre[(7)] defines the *Bogoliubov approximation* for a Hamiltonian H_Λ in $\mathscr{F}_\Lambda$ as follows:

Definition A.1. The Bogoliubov approximation $H_\Lambda\ (c^\#, \mu)$ for a Hamiltonian $H_\Lambda(\mu) \equiv H_\Lambda - \mu N_\Lambda$ on $\mathscr{F}_\Lambda \approx \mathscr{F}_{0\Lambda} \otimes \mathscr{F}'_\Lambda$ is the operator defined on $\mathscr{F}'_\Lambda$ by its quadratic form

$$(\psi'_1, H_\Lambda(c^\#, \mu)\, \psi'_2)_{\mathscr{F}'_\Lambda} \equiv (\psi_{0\Lambda}(c) \otimes \psi'_1, H_\Lambda(\mu)\, \psi_{0\Lambda}(c) \otimes \psi'_2)_{\mathscr{F}_\Lambda}$$

for $\psi_{0\Lambda}(c) \otimes \psi'_{1,2}$ in the form-domain of $H_\Lambda(\mu)$, where $c^\# = (c \text{ or } \bar{c})$.

A.2. Application to the WIBG

Notice that the self-adjoint operator H^B_Λ is defined on a dense domain in the boson Fock space $\mathscr{F}_\Lambda \approx \mathscr{F}_{0\Lambda} \otimes \mathscr{F}'_\Lambda$ over $L^2(\Lambda)$. Then by definition A.1, the *Bogoliubov approximation* for the Bogoliubov Hamiltonian (1.4) gets the form:

$$\begin{aligned} H^B_\Lambda(c^\#, \mu) = & \sum_{k \in \Lambda^*, k \neq 0} [\varepsilon_k - \mu + v(0)\, |c|^2]\, a_k^* a_k \\ & + \tfrac{1}{2} \sum_{k \in \Lambda^*, k \neq 0} v(k)\, |c|^2\, [a_k^* a_k + a_{-k}^* a_{-k}] \\ & + \tfrac{1}{2} \sum_{k \in \Lambda^*, k \neq 0} v(k) [c^2 a_k^* a_{-k}^* + \bar{c}^2 a_k a_{-k}] \\ & - \mu\, |c|^2\, V + \tfrac{1}{2} v(0)\, |c|^4\, V \end{aligned} \tag{A.3}$$

The Hamiltonian (A.3) can be diagonalized by the Bogoliubov u–v transformation.[1, 2] In contrast to H_Λ^B the pressure calculated with $H_\Lambda^B(c^\#, \mu)$:

$$\tilde{p}_\Lambda^B(\beta, \mu; c^\#) = \frac{1}{\beta V} \ln Tr_{\mathscr{F}'_\Lambda} e^{-\beta H_\Lambda^B(c^\#, \mu)} \tag{A.4}$$

is well-defined instead of $\mu \leqslant 0$ for all $\mu \leqslant v(0)\, |c|^2$. It has the following explicit form:

$$\tilde{p}_\Lambda^B(\beta, \mu; c^\#) = \xi_\Lambda(\beta, \mu; x) + \eta_\Lambda(\mu; x)$$

$$\xi_\Lambda(\beta, \mu; x) = \frac{1}{\beta V} \sum_{k \in \Lambda^*, k \neq 0} \ln(1 - e^{-\beta E_k})^{-1} \tag{A.5}$$

$$\eta_\Lambda(\mu; x) = -\frac{1}{2V} \sum_{k \in \Lambda^*, k \neq 0} (E_k - f_k) + \mu x - \frac{1}{2} v(0)\, x^2$$

with $x = |c|^2 \geqslant 0$ and

$$\begin{aligned} f_k &= \varepsilon_k - \mu + x[v(0) + v(k)] \\ h_k &= x v(k) \\ E_k &= \sqrt{f_k^2 - h_k^2} \end{aligned} \tag{A.6}$$

Notice that the spectrum E_k is gapless if one puts $\mu = v(0)\, |c|^2 > 0$,[1, 2] which is out of the stability domain $Q = \{\mu \leqslant 0\} \times \{\theta \geqslant 0\}$ for the Bogoliubov WIBG.

APPENDIX B. CLASSIFICATION OF BOSE CONDENSATIONS

B.1. The van den Berg–Lewis–Pulè Classification (Condensations of Type I, II and III)

For reader's convenience we remind a nomenclature of (generalized) Bose condensations according to ref. 12–14:

— a condensation is called *type I* when a finite number of single-particle levels are macroscopically occupied;

— it is of *type II* when an infinite number of the levels are macroscopically occupied;

— it is called *type III*, or the *non-extensive* condensation, when no of the levels are macroscopically occupied whereas one has

$$\lim_{\delta \to 0^+} \lim_{\Lambda} \frac{1}{V} \sum_{\{k \in \Lambda^*,\, 0 \leqslant \|k\| \leqslant \delta\}} \langle N_k \rangle = \rho - \rho_c(\theta)$$

An example of these different condensations is given in ref. 12. This paper demonstrates that three types of Bose–Einstein condensation can be realized in the case of the PBG in an anisotropic rectangular box $\Lambda \subset \mathbb{R}^3$ of volume $V = |\Lambda| = L_x.L_y.L_z$ and with Dirichlet boundary conditions. Let $L_x = V^{\alpha_x}$, $L_y = V^{\alpha_y}$, $L_z = V^{\alpha_z}$ for $\alpha_x + \alpha_y + \alpha_z = 1$ and $\alpha_x \leqslant \alpha_y \leqslant \alpha_z$. If $\alpha_z < 1/2$, then for sufficiently large density ρ, we have the Bose–Einstein condensation of type I in the fundamental mode $k = (2\pi/L_x, 2\pi/L_y, 2\pi/L_z)$. For $\alpha_z = 1/2$ one gets a condensation of type II characterized by a macroscopic occupation of infinite package of modes $k = (2\pi/L_x, 2\pi/L_y, 2\pi n/L_z)$, $n \in \mathbb{N}$, whereas for $\alpha_z > 1/2$ we obtain a condensation of type III. In ref. 28, 29 it was shown that this type III condensation can be caused in the PBG by a weak external potential or (see ref. 13, 30) by a specific choice of boundary conditions and geometry. Another example of the *non-extensive* condensation is given in ref. 10, 11 for bosons in an *isotropic* box Λ with *repulsive interactions* which spread out the *conventional* Bose–Einstein condensation of type I into Bose–Einstein condensation of type III.

B.2. Nonconventional Versus Conventional Bose Condensation

Here we classify Bose condensations by their mechanisms of formation. In the most of papers (cf. refs. 10, 12–14, 28–30), the condensation is due to a *saturation* of the total particle density, originally discovered by Einstein[31] in the Bose gas without interaction (PBG). We call it the *conventional* Bose–Einstein condensation.[32]

The existence of a new kind of condensation, which is induced by *interaction*, is pointed out in recent papers.[4–6, 11] In particular, this is the case of the Bogoliubov Weakly Imperfect Bose Gas. We call it the *nonconventional* Bose condensation.

(i) As it is shown in the present paper (see also ref. 11), the nonconventional condensation does not exclude the appearance of the Bose–Einstein condensation when total density of particles grows and exceeds some saturation limit $\rho_c^B(\theta)$.

(ii) To appreciate the notion of nonconventional condensation let us remark that in models (4.2) and (4.3)[11] for $d = 1, 2$, there exists only one kind of condensation, namely the nonconventional. What concerning the WIBG, see Remark 4.1.

Since the known Bose-systems manifesting condensations are far from to be perfect, the concept of condensation induced by interaction is rather natural. For example in a condensate of sodium atoms interaction seems to predominate compare with kinetic energy.[33] Therefore, condensation in trapped alkali dilute-gases,[33–35] should be a combination of *nonconventional* and *conventional* Bose condensations.

Remark B.1. A nonconventional Bose condensation can always be characterized by its type. Therefore, formally one obtains six kinds of condensations: a nonconventional versus conventional of types I, II, or III.

APPENDIX C. THE GRIFFITHS LEMMA[8, 9]

Lemma C.1. Let $\{f_n(x)\}_{n\geqslant 1}$ be a sequence of convex functions on a compact $I\subset\mathbb{R}$. If there exists a pointwise limit

$$\lim_{n\to\infty} f_n(x)=f(x), \qquad x\in I \tag{C.1}$$

then

$$\begin{aligned}\lim_{n\to\infty}\inf \partial_x f_n(x-0)\geqslant \partial_x f(x-0)\\ \lim_{n\to\infty}\sup \partial_x f_n(x+0)\leqslant \partial_x f(x+0)\end{aligned} \tag{C.2}$$

Proof. By convexity one has

$$\begin{aligned}\partial_x f_n(x+0)\leqslant\frac{1}{l}[f_n(x+l)-f_n(x)]\\ \partial_x f_n(x-0)\geqslant\frac{1}{l}[f_n(x)-f_n(x-l)]\end{aligned} \tag{C.3}$$

for $l>0$. Then taking the limit $n\to\infty$ in (C.3), by (C.1) we obtain:

$$\begin{aligned}\lim_{n\to\infty}\sup \partial_x f_n(x+0)\leqslant\frac{1}{l}[f(x+l)-f(x)]\\ \lim_{n\to\infty}\inf \partial_x f_n(x-0)\geqslant\frac{1}{l}[f(x)-f(x-l)]\end{aligned} \tag{C.4}$$

Now taking the limit $l\to+0$, in (C.4), one gets (C.2). ∎

Remark C.2. In particular, if $x_0 \in I$ is such that $\partial_x f_n(x_0 - 0) = \partial_x f_n(x_0 + 0)$ and $\partial_x f(x_0 - 0) = \partial_x f(x_0 + 0)$, then

$$\lim_{n \to \infty} \partial_x f_n(x_0) = \partial_x f(x_0)$$

APPENDIX D

Lemma D.1. Let the interaction potential (1.1) satisfies (A) and (B). Then there exists $\varepsilon_{\Lambda, 1}$:

$$\varepsilon_{\Lambda, 1} \in \left[\inf_{k \neq 0} \left(\varepsilon_k - \frac{v(k)}{2V} \right), \hat{\varepsilon}_{\Lambda, 1} = \inf_{k \neq 0} \varepsilon_k = \varepsilon_{\|k\| = 2\pi/L} \right]$$

such that for $\mu < \varepsilon_{\Lambda, 1}$

$$\begin{aligned} p_\Lambda^B(\beta, \mu) &< +\infty \\ \omega_\Lambda^B \left(\frac{N_\Lambda}{V} \right) &< +\infty \end{aligned} \tag{D.1}$$

and

$$\begin{aligned} \lim_{\mu \to \varepsilon_{\Lambda, 1}} p_\Lambda^B(\beta, \mu) &= +\infty \\ \lim_{\mu \to \varepsilon_{\Lambda, 1}} \omega_\Lambda^B \left(\frac{N_\Lambda}{V} \right) &= +\infty \end{aligned} \tag{D.2}$$

Proof. Since $v(k)$ satisfies (A) and (B), by regrouping terms in (1.6), (1.7) one gets

$$\begin{aligned} H_\Lambda^B = \tilde{H}_\Lambda &+ \frac{v(0)}{V} a_0^* a_0 \sum_{\{k \in \Lambda^*, k \neq 0\}} a_k^* a_k \\ &+ \frac{1}{2V} \sum_{\{k \in \Lambda^*, k \neq 0\}} v(k) (a_0^* a_k + a_{-k}^* a_0)^* (a_0^* a_k + a_{-k}^* a_0) \end{aligned} \tag{D.3}$$

where

$$\tilde{H}_\Lambda = \sum_{\{k \in \Lambda^*, k \neq 0\}} \left(\varepsilon_k - \frac{v(k)}{2V} \right) a_k^* a_k + \frac{v(0)}{2V} (a_0^* a_0)^2 - \frac{1}{2} \varphi(0) a_0^* a_0 \tag{D.4}$$

Then from (D.3), (D.4) we obtain

$$H_{\Lambda}^{B} \geqslant \tilde{H}_{\Lambda} \tag{D.5}$$

By straightforward calculations one gets

$$p_{\Lambda}[\tilde{H}_{\Lambda}] = \frac{1}{\beta V} \sum_{k \in \Lambda^*, k \neq 0} \ln\{1 - e^{-\beta[\varepsilon_k - (\mu + (v(k)/2V))]}\}^{-1} + \frac{1}{\beta V} \ln \sum_{n_0 = 0}^{+\infty} e^{\beta V[(\mu + 1/2\varphi(0))(n_0/V) - (v(0)/2V)(n_0/V)^2]}$$

which together with (D.5) implies

$$p_{\Lambda}^{B}(\beta, \mu) \equiv p_{\Lambda}[H_{\Lambda}^{B}] \leqslant p_{\Lambda}[\tilde{H}_{\Lambda}] < +\infty \tag{D.6}$$

for $\mu < \inf_{k \neq 0}(\varepsilon_k - (v(k)/2V))$. Since

$$\omega_{\Lambda}^{B}\left(\frac{N_{\Lambda}}{V}\right) = \partial_{\mu} p_{\Lambda}^{B}(\beta, \mu)$$

by (D.6) and by convexity of the pressure $p_{\Lambda}^{B}(\beta, \mu)$ as a function of μ we deduce that

$$\omega_{\Lambda}^{B}\left(\frac{N_{\Lambda}}{V}\right) < +\infty$$

for $\mu < \inf_{k \neq 0}(\varepsilon_k - (v(k)/2V))$. Moreover, by the Peierls–Bogoliubov inequality (see e.g., refs. 36, 37), one gets:

$$\frac{1}{V} \langle U_{\Lambda} \rangle_{H_{\Lambda}^{B}} \leqslant p_{\Lambda}[H_{\Lambda}^{BD}] - p_{\Lambda}[H_{\Lambda}^{B}] \leqslant \frac{1}{V} \langle U_{\Lambda} \rangle_{H_{\Lambda}^{BD}} \tag{D.7}$$

where $H_{\Lambda}^{BD} \equiv T_{\Lambda} + U_{\Lambda}^{D}$ is a diagonal part of the Bogoliubov Hamiltonian with T_{Λ} and U_{Λ}^{D} defined respectively by (1.5) and (1.6). Since $\langle U_{\Lambda} \rangle_{H_{\Lambda}^{BD}} = 0$, we deduce from (D.7) that

$$p_{\Lambda}^{B}(\beta, \mu) \geqslant p_{\Lambda}[H_{\Lambda}^{BD}]$$

Combining this inequality with the estimate (cf. ref. 5)

$$p_{\Lambda}[H_{\Lambda}^{BD}] \geqslant \frac{1}{\beta V} \sum_{k \in \Lambda^*, k \neq 0} \ln[(1 - e^{[-\beta(\varepsilon_k - \mu)]})^{-1}]$$

we get

$$\lim_{\substack{\mu \to \inf \varepsilon_k \\ k \neq 0}} p_\Lambda[H_\Lambda^{BD}] = +\infty \tag{D.8}$$

Therefore, by (D.6) and (D.8) we find that there exists $\varepsilon_{\Lambda,1} \in [\inf_{k \neq 0}(\varepsilon_k - (v(k)/2V)), \inf_{k \neq 0} \varepsilon_k]$ such that $p_\Lambda^B(\beta, \mu)$ and $\omega_\Lambda^B(N_\Lambda/V)$ are bounded for $\mu < \varepsilon_{\Lambda,1}$ and

$$\lim_{\mu \to \varepsilon_{\Lambda,1}} p_\Lambda^B(\beta, \mu) = +\infty \tag{D.9}$$

Notice that by convexity of $p_\Lambda^B(\beta, \mu)$ one gets

$$\frac{p_\Lambda^B(\beta, \mu) - p_\Lambda^B(\beta, 0)}{\mu} \leqslant \partial_\mu p_\Lambda^B(\beta, \mu) = \omega_\Lambda^B\left(\frac{N_\Lambda}{V}\right)$$

Then the limit (D.9) implies

$$\lim_{\mu \to \varepsilon_{\Lambda,1}} \omega_\Lambda^B\left(\frac{N_\Lambda}{V}\right) = +\infty$$

which completes the proof of (D.2). ∎

Lemma D.2. Let $\|k\| > 2\pi/L$. Then for the Gibbs state $\omega_{\Lambda,\rho}^B(-)$ we have:

$$\omega_{\Lambda,\rho}^B(N_k) \leqslant \frac{1}{e^{B_k(\mu_\Lambda^B(\theta,\rho))} - 1} + \beta \frac{v(k)}{2V} \frac{\omega_{\Lambda,\rho}^B(a_0^* a_0)}{1 - e^{-B_k(\mu_\Lambda^B(\theta,\rho))}} \tag{D.10}$$

with

$$B_k(\mu = \mu_\Lambda^B(\theta, \rho)) \equiv \beta\left[\varepsilon_k - \mu_\Lambda^B(\theta, \rho) - \frac{v(k)}{2V}\right] \tag{D.11}$$

Proof. By the Fannes–Verbeure correlation inequalities for the Gibbs state $\omega_\Lambda^B(-) \equiv \langle - \rangle_{H_\Lambda^B}(\beta, \mu)$ (see refs. 22, 38, 39):

$$\beta\omega_\Lambda^B(X^*[H_\Lambda^B(\mu), X]) \geqslant \omega_\Lambda^B(X^*X) \ln \frac{\omega_\Lambda^B(X^*X)}{\omega_\Lambda^B(XX^*)} \tag{D.12}$$

where X is an observable from domain of the commutator $[H_\Lambda^B(\mu), .]$, we obtain

$$\beta\omega_\Lambda^B(a_k^*[H_\Lambda^B(\mu), a_k]) \geqslant \omega_\Lambda^B(N_k) \ln \frac{\omega_\Lambda^B(N_k)}{\omega_\Lambda^B(N_k)+1} \tag{D.13}$$

for $X = a_k$. Since for $\|k\| > 2\pi/L$

$$[H_\Lambda^B(\mu), a_k] = -\left(\varepsilon_k - \mu - [v(0)+v(k)]\frac{a_0^* a_0}{V}\right) a_k - \frac{v(k)}{V} a_0^2 a_{-k}^*$$

one gets for $\mu = \mu_\Lambda^B(\theta, \rho)$ that

$$\begin{aligned}\omega_{\Lambda,\rho}^B(a_k^*[H_\Lambda^B(\mu_\Lambda^B(\theta,\rho)), a_k]) = &-[\varepsilon_k - \mu_\Lambda^B(\theta,\rho)]\,\omega_{\Lambda,\rho}^B(N_k)\\ &-[v(0)+v(k)]\frac{\omega_{\Lambda,\rho}^B(a_0^* a_0 N_k)}{V}\\ &-v(k)\frac{\omega_{\Lambda,\rho}^B(a_0^2 a_k^* a_{-k}^*)}{V}\end{aligned} \tag{D.14}$$

Notice that $\omega_{\Lambda,\rho}^B(a_k^*[H_\Lambda^B(\mu_\Lambda^B(\theta,\rho)), a_k]) \in \mathbb{R}$, then by (D.14) $\omega_{\Lambda,\rho}^B(a_0^2 a_k^* a_{-k}^*) \in \mathbb{R}$. Therefore,

$$2\omega_{\Lambda,\rho}^B(a_0^2 a_k^* a_{-k}^*) = \omega_{\Lambda,\rho}^B(a_0^2 a_k^* a_{-k}^*) + \omega_{\Lambda,\rho}^B(a_k a_{-k} a_0^{*2}) \tag{D.15}$$

Moreover, since the functions ε_k and $v(k)$ are even, we have

$$\omega_{\Lambda,\rho}^B(a_0^* a_0 N_k) = \omega_{\Lambda,\rho}^B(a_0^* a_0 N_{-k}) \tag{D.16}$$

Thus (D.14)–(D.16) imply

$$\begin{aligned}\omega_{\Lambda,\rho}^B(a_k^*[H_\Lambda^B(\mu_\Lambda^B(\theta,\rho)), a_k]) = &-[\varepsilon_k - \mu_\Lambda^B(\theta,\rho)]\,\omega_{\Lambda,\rho}^B(a_k^* a_k)\\ &-\frac{v(k)}{2V}\omega_{\Lambda,\rho}^B(a_0^2 a_k^* a_{-k}^* + a_0^{*2} a_k a_{-k})\\ &-\frac{[v(0)+v(k)]}{2V}\omega_{\Lambda,\rho}^B(a_0^* a_0 N_k + a_0^* a_0 N_{-k})\end{aligned} \tag{D.17}$$

Now by identity

$$\begin{aligned}&a_0^2 a_k^* a_{-k}^* + a_0^{*2} a_{-k} a_k + a_0^* a_0 a_k^* a_k + a_0^* a_0 a_{-k}^* a_{-k}\\ &\quad = (a_0^* a_k + a_{-k}^* a_0)^* (a_0^* a_k + a_{-k}^* a_0) - a_k^* a_k - a_0^* a_0\end{aligned} \tag{D.18}$$

we deduce from (D.17) the estimate:

$$\omega^B_{\Lambda,\beta}(a^*_k[H^B_\Lambda(\mu^B_\Lambda(\theta,\rho)),a_k]) \leqslant -\left[\varepsilon_k-\mu^B_\Lambda(\theta,\rho)-\frac{v(k)}{2V}\right]\omega^B_{\Lambda,\rho}(N_k)+\frac{v(k)}{2V}\,\omega^B_{\Lambda,\rho}(a^*_0 a_0) \tag{D.19}$$

Therefore, combining (D.13) with (D.19) we find:

$$B_k(\mu^B_\Lambda(\theta,\rho))\,\omega^B_{\Lambda,\rho}(N_k)-\beta\frac{v(k)}{2V}\,\omega^B_{\Lambda,\rho}(a^*_0 a_0)\leqslant\omega^B_{\Lambda,\rho}(N_k)\ln\frac{\omega^B_{\Lambda,\rho}(N_k)+1}{\omega^B_{\Lambda,\rho}(N_k)} \tag{D.20}$$

with $B_k(\mu^B_\Lambda(\theta,\rho))$ defined by (D.11). Since

$$\mu^B_\Lambda(\theta,\rho)<\varepsilon_{\Lambda,1}<\varepsilon_{\|k\|=2\pi/L}=\inf_{k\neq 0}\varepsilon_k$$

and $\|k\|>2\pi/L$, one has $B_k(\mu^B_\Lambda(\theta,\rho))>0$. Hence, to estimate $x\equiv\omega^B_{\Lambda,\rho}(N_k)$, we have to solve the inequality

$$B_k(\mu^B_\Lambda(\theta,\rho))\,x-\beta\frac{v(k)}{2V}\,\omega^B_{\Lambda,\rho}(a^*_0 a_0)\leqslant x\ln\frac{x+1}{x} \tag{D.21}$$

for $x\geqslant 0$. Notice that the solution of (D.21) is the set $\{0\leqslant x\leqslant x_1\}$, where x_1 is a solution of the equation

$$B_k(\mu^B_\Lambda(\theta,\rho))\,x_1-\beta\frac{v(k)}{2V}\,\omega^B_{\Lambda,\rho}(a^*_0 a_0)=x_1\ln\frac{x_1+1}{x_1}$$

Let

$$x_2=\frac{1}{e^{B_k(\mu^B_\Lambda(\theta,\rho))}-1} \tag{D.22}$$

be a nontrivial solution of the equation

$$B_k(\mu^B_\Lambda(\theta,\rho))\,x=x\ln\frac{x+1}{x}$$

Then the inequality $x\leqslant x_1$ can be rewritten as

$$x\leqslant x_2+(x_1-x_2) \tag{D.23}$$

Since the function $f(x) \equiv x \ln(1+1/x)$ defined for $x \geqslant 0$ is concave, we have

$$\frac{f(x_1)-f(x_2)}{f'(x_2)} \leqslant x_1 - x_2$$

from which by (D.22), (D.23) one gets (D.10) for $\|k\| > 2\pi/L$. ▮

APPENDIX E

Lemma E.1. For any $\mu \leqslant 0$, the pressure $\hat{p}^B(\beta,\mu) \equiv \lim_\Lambda \hat{p}^B_\Lambda(\beta,\mu)$, see (3.2), is given by

$$\hat{p}^B(\beta,\mu) = \sup_{c \in \mathbb{C}} \tilde{p}^B(\beta,\mu; c^{\#}) = p^B(\beta,\mu) \tag{E.1}$$

Proof. First we remark that by virtue of (A.5) and (A.6) there is $B_1 > 0$ such that

$$\tilde{p}^B_\Lambda(\beta,\mu; c^{\#}) \leqslant B_1 - \tfrac{1}{2} v(0)\, |c|^4 \tag{E.2}$$

Then, the optimal value of $|c|^2$ for $\sup_{c\in\mathbb{C}} \tilde{p}^B(\beta,\mu; c^{\#})$ is bounded by a positive constant $M < \infty$ which can be chosen in such a way that, for $|c|^2 \geqslant M$ and V sufficiently large, one has

$$\tilde{p}^B_\Lambda(\beta,\mu; c^{\#}) \leqslant -B_2\, |c|^2 \tag{E.3}$$

for some $B_2 > 0$. We put for short

$$\tilde{p}^B_\Lambda(\mu, x) \equiv \tilde{p}^B_\Lambda(\beta,\mu; c^{\#}) \tag{E.4}$$

where $x = |c|^2$, see (A.5). Then

$$\tilde{p}^B(\mu, x) \equiv \lim_\Lambda \tilde{p}^B_\Lambda(\mu, x) \tag{E.5}$$

Therefore, by (E.2) there exists $M < \infty$ such that

$$\sup_{x\in\mathbb{R}^+} \tilde{p}^B_\Lambda(\mu, x) = \sup_{x\in[0,M]} \tilde{p}^B_\Lambda(\mu, x) = \tilde{p}^B_\Lambda(\mu, \hat{x}_\Lambda < M) \tag{E.6}$$

and consequently

$$\sup_{x\in\mathbb{R}^+} \tilde{p}^B(\mu, x) = \tilde{p}^B(\mu, \hat{x} < M) \tag{E.7}$$

with $\hat{x} \equiv \lim_{\Lambda} \hat{x}_{\Lambda}$. Since by (3.1) one has

$$\hat{p}^{B}_{\Lambda}(\beta, \mu) \geqslant \frac{1}{\beta V} \ln \int_{0}^{M} e^{\beta V \tilde{p}^{B}_{\Lambda}(\mu, x)} \, dx \tag{E.8}$$

and since by (E.3) we get

$$\hat{p}^{B}_{\Lambda}(\beta, \mu) \leqslant \frac{1}{\beta V} \ln \int_{0}^{M} e^{\beta V \tilde{p}^{B}_{\Lambda}(\mu, x)} \, dx + \frac{1}{\beta V} \ln \int_{M}^{+\infty} e^{-\beta V B_2 x} \, dx \tag{E.9}$$

the standard large deviation arguments (see e.g., refs. 16, 17) and (E.7) imply

$$\hat{p}^{B}(\beta, \mu) = \sup_{x \in \mathbb{R}^+} \tilde{p}^{B}(\mu, x)$$

Therefore from (A.5), (E.5) and Proposition 1.1 we obtain (E.1). ∎

Lemma E.2. Let

$$\widetilde{\mathbb{K}}^{\mu}_{\Lambda}[dx] \equiv e^{\beta V [\tilde{p}^{B}_{\Lambda}(\mu, x) - \hat{p}^{B}_{\Lambda}(\beta, \mu)]} \, dx \tag{E.10}$$

with $\tilde{p}^{B}_{\Lambda}(\mu, x)$ defined by (E.4). Then:

(i) for $\mu_0(\theta) < \mu \leqslant 0$ one has

$$\widetilde{\mathbb{K}}^{\mu}[dx] \equiv \lim_{\Lambda} \widetilde{\mathbb{K}}^{\mu}_{\Lambda}[dx] = \delta(x - |\hat{c}(\theta, \mu)|^2) \, dx \tag{E.11}$$

(ii) whereas for $\mu < \mu_0(\theta)$ we get

$$\widetilde{\mathbb{K}}^{\mu}[dx] \equiv \lim_{\Lambda} \widetilde{\mathbb{K}}^{\mu}_{\Lambda}[dx] = \delta(x) \, dx \tag{E.12}$$

Proof. (i) For $\mu_0(\theta) < \mu \leqslant 0$, one has

$$\sup_{x \geqslant 0} \tilde{p}^{B}_{\Lambda}(\mu, x) = \tilde{p}^{B}_{\Lambda}(\mu, \hat{x}_{\Lambda})$$

whereas

$$\sup_{x \geqslant 0} \tilde{p}^{B}(\mu, x) = \tilde{p}^{B}(\mu, \hat{x})$$

where

$$\hat{x} = \lim_{\Lambda} \hat{x}_{\Lambda} < M$$

For any positive continuous function $f(x)$ defined on $\mathbb{R}^+$ and such that

$$\int_0^{+\infty} f(x)\, e^{\beta V \tilde{p}_\Lambda^B(\mu, x)}\, dx < +\infty \tag{E.13}$$

we have

$$\int_{\mathbb{R}^+} f(x)\, \tilde{\mathbb{K}}_\Lambda^\mu[dx] = e^{-\beta V \hat{p}_\Lambda^B(\beta, \mu)} \int_{\mathbb{R}^+} f(x)\, e^{\beta V \tilde{p}_\Lambda^B(\mu, x)}\, dx$$

Then for each $\varepsilon > 0$:

$$\begin{aligned} \int_{\mathbb{R}^+} f(x)\, \tilde{\mathbb{K}}_\Lambda^\mu[dx] = {} & e^{-\beta V \hat{p}_\Lambda^B(\beta, \mu)} f(x_\varepsilon) \int_{\hat{x}-\varepsilon}^{\hat{x}+\varepsilon} e^{\beta V \tilde{p}_\Lambda^B(\mu, x)}\, dx \\ & + e^{-\beta V \hat{p}_\Lambda^B(\beta, \mu)} f(x_\varepsilon^-) \int_0^{\hat{x}-\varepsilon} e^{\beta V \tilde{p}_\Lambda^B(\mu, x)}\, dx \\ & + e^{-\beta V \hat{p}_\Lambda^B(\beta, \mu)} f(x_\varepsilon^+) \int_{\hat{x}+\varepsilon}^{M} e^{\beta V \tilde{p}_\Lambda^B(\mu, x)}\, dx \\ & + e^{-\beta V \hat{p}_\Lambda^B(\beta, \mu)} \int_M^{+\infty} f(x)\, e^{\beta V \tilde{p}_\Lambda^B(\mu, x)}\, dx \end{aligned} \tag{E.14}$$

where $x_\varepsilon \in [\hat{x}-\varepsilon, \hat{x}+\varepsilon]$, $x_\varepsilon^- \in [0, \hat{x}-\varepsilon]$, and $x_\varepsilon^+ \in [\hat{x}+\varepsilon, M]$. Combining the standard Laplace large deviation principle[16, 17] and (3.2), we get

$$\lim_\Lambda \left[\frac{1}{\beta V} \ln \int_a^b e^{\beta V \tilde{p}_\Lambda^B(\mu, x)}\, dx - \hat{p}_\Lambda^B(\beta, \mu) \right] = \sup_{x \in (a, b)} \tilde{p}^B(\mu, x) - \sup_{x \in \mathbb{R}^+} \tilde{p}^B(\mu, x)$$

for any $a, b \in \mathbb{R}^+$ and thus

$$\lim_\Lambda \left[e^{-\beta V \hat{p}_\Lambda^B(\beta, \mu)} \int_a^b e^{\beta V \tilde{p}_\Lambda^B(\mu, x)}\, dx \right] = \chi_{(a, b)}(x = \hat{x}) \tag{E.15}$$

where $\chi_{(a, b)}(x)$ is the characteristic function of (a, b). Since by (E.3) one has

$$\int_M^{+\infty} f(x)\, e^{\beta V \tilde{p}_\Lambda^B(\mu, x)}\, dx \leqslant \int_M^{+\infty} f(x)\, e^{-\beta V B_2 x}\, dx$$

one deduces from (E.14) and (E.15) that

$$\lim_{\varepsilon \to 0^+} \lim_\Lambda \int_{\mathbb{R}^+} f(x)\, \tilde{\mathbb{K}}_\Lambda^\mu[dx] = \lim_{\varepsilon \to 0^+} f(x_\varepsilon) = f(\hat{x})$$

The proof of (ii) is the same as for (i) but with a supremum of $\tilde{p}^B(\mu, x)$ at $x=0$. ▮

Lemma E.3. For $\mu_\Lambda = \mu_0(\theta) + \gamma/\beta V$ with $\gamma \in \mathbb{R}$ we have

$$\lim_\Lambda \widetilde{\mathbb{K}}^{\mu_\Lambda}_\Lambda[dx] = \{\lambda_\gamma\, \delta(x) + (1-\lambda_\gamma)\, \delta(x - |\hat{c}(\theta, \mu)|^2)\}\, dx \tag{E.16}$$

where λ_γ is defined by

$$\lambda_\gamma \equiv \frac{e^{\gamma \rho^B_{\inf}(\theta)}}{e^{\gamma \rho^B_{\inf}(\theta)} + e^{\gamma \rho^B_{\sup}(\theta)}} \in [0, 1] \tag{E.17}$$

Proof. For $\mu = \mu_0(\theta) \leqslant 0$, the function $\tilde{p}^B_\Lambda(\mu, x)$ (E.4) has a degenerate supremum at $\hat{x}_\Lambda > 0$ and at $x = 0$, which implies that in the thermodynamic limit

$$\sup_{x \geqslant 0} \tilde{p}^B(\mu_0(\theta), x) = \tilde{p}^B(\mu_0(\theta), \hat{x}) = \tilde{p}^B(\mu_0(\theta), x=0)$$

where $\hat{x} < M$. Since

$$\tilde{p}^B_\Lambda(\mu_\Lambda, x) = \tilde{p}^B_\Lambda(\mu_0(\theta), x) + \frac{\gamma}{\beta V}\, \partial_\mu \tilde{p}^B_\Lambda(\mu_0(\theta), x) + O\left(\frac{1}{V^2}\right)$$

for any positive continuous function $f(x)$ defined on $\mathbb{R}^+$ and satisfying (E.13) we get

$$\int_{\mathbb{R}^+} f(x)\, \widetilde{\mathbb{K}}^{\mu_{0\Lambda}(\theta)}_\Lambda[dx] = \frac{\int_{\mathbb{R}^+} [f(x)\, e^{\gamma\, \partial_\mu \tilde{p}^B_\Lambda(\mu_0(\theta), x) + O(1/V)}]\, e^{\beta V \tilde{p}^B_\Lambda(\mu_0(\theta), x)}\, dx}{\int_{\mathbb{R}^+} [e^{\gamma\, \partial_\mu \tilde{p}^B_\Lambda(\mu_0(\theta), x) + O(1/V)}]\, e^{\beta V \tilde{p}^B_\Lambda(\mu_0(\theta), x)}\, dx} \tag{E.18}$$

By the Laplace large deviation principle[16, 17] together with Lemma E.1 we find that

$$\lim_\Lambda e^{-\beta V \tilde{p}^B_\Lambda(\beta, \mu_0(\theta))} \int_{\mathbb{R}^+} h_\Lambda(x)\, e^{\beta V \tilde{p}^B_\Lambda(\mu_0(\theta), x)}\, dx = h(0) + h(\hat{x}) \tag{E.19}$$

Here functions

$$h(x) = \lim_\Lambda h_\Lambda(x)$$

are defined on $\mathbb{R}^+$ and satisfy (E.13). Since

$$\lim_{\Lambda} \partial_\mu \tilde{p}_\Lambda^B(\mu_\Lambda, 0) = \rho_{\inf}^B(\theta)$$

$$\lim_{\Lambda} \partial_\mu \tilde{p}_\Lambda^B(\mu_\Lambda, \hat{x}_\Lambda) = \rho_{\sup}^B(\theta)$$

(cf. (2.5)), by (E.19) for $h_\Lambda(x) = f(x)^{\gamma\, \partial_\mu \tilde{p}_\Lambda^B(\mu_0(\theta), x) + O(1/V)}$ (numerator) and for $h_\Lambda(x) = e^{\gamma\, \partial_\mu \tilde{p}_\Lambda^B(\mu_0(\theta), x) + O(1/V)}$ (denominator), the fraction (E.18) implies (E.16) and (E.17) in the thermodynamic limit. ▮

ACKNOWLEDGMENT

We benefited a lot from very helpful discussions concerning the WIBG with Nicolae Angelescu, Tom Michoel and André Verbeure. We wish to thank the Intituut voor Theoretische Fysica of K.U. Leuven for hospitality during our visit when the main ideas of this paper were formulated.

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